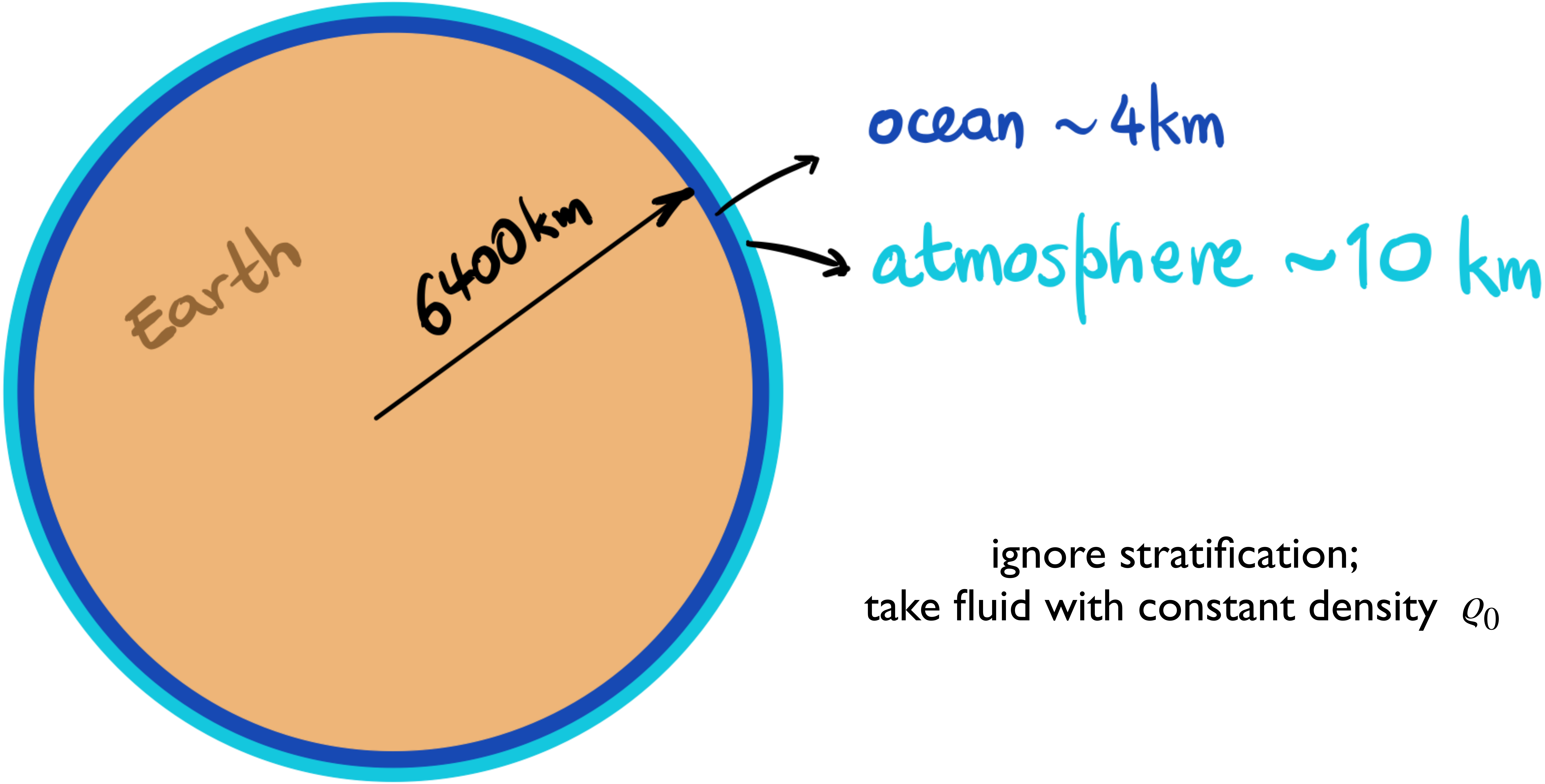
A taste of Quasi-Geostrophy

CLEX Winter School 2020
Atmosphere & Ocean Dynamics

(Teaser version via zoom)

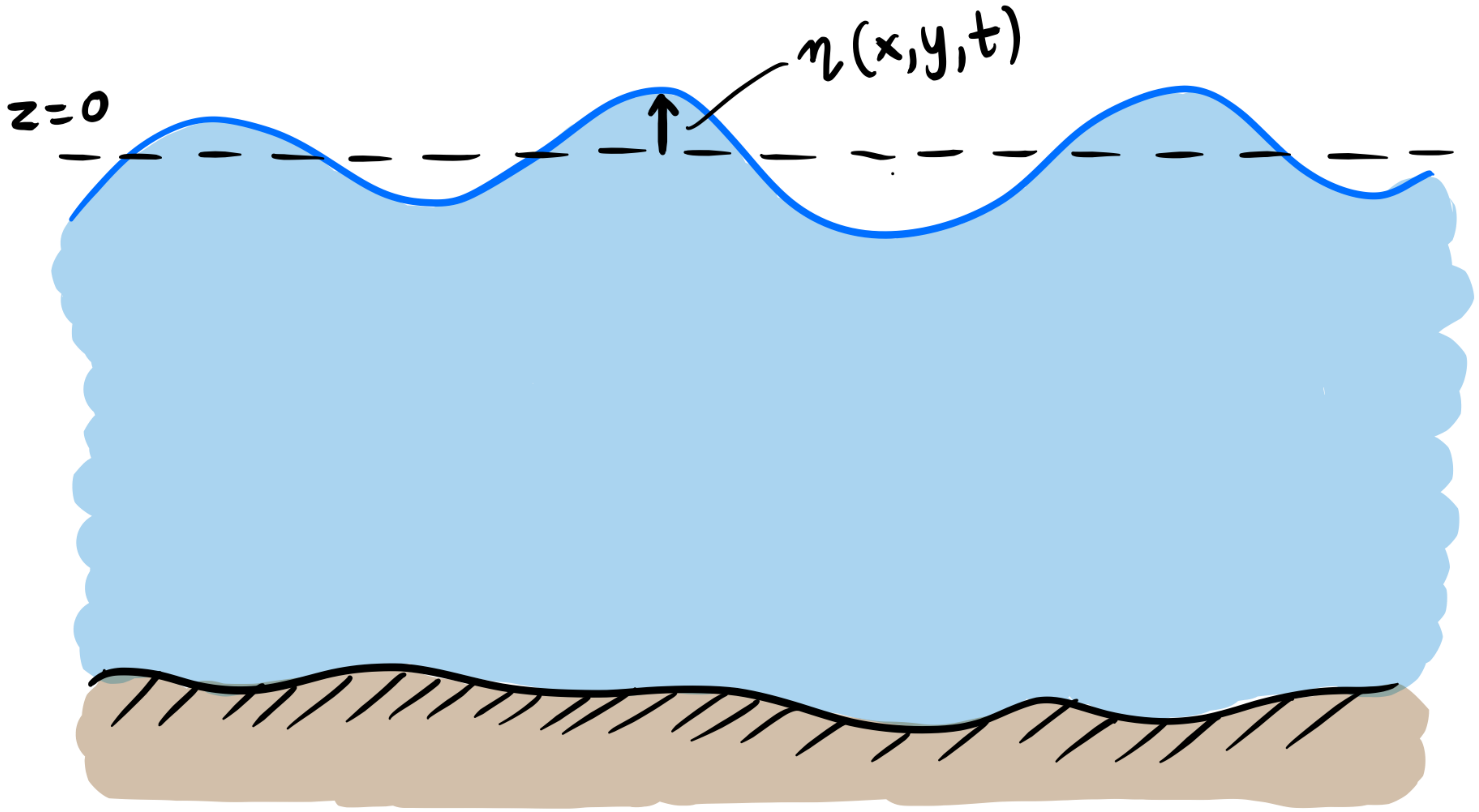
Navid Constantinou
ANU

Shallow-water dynamics



ignore stratification;
take fluid with constant density ϱ_0

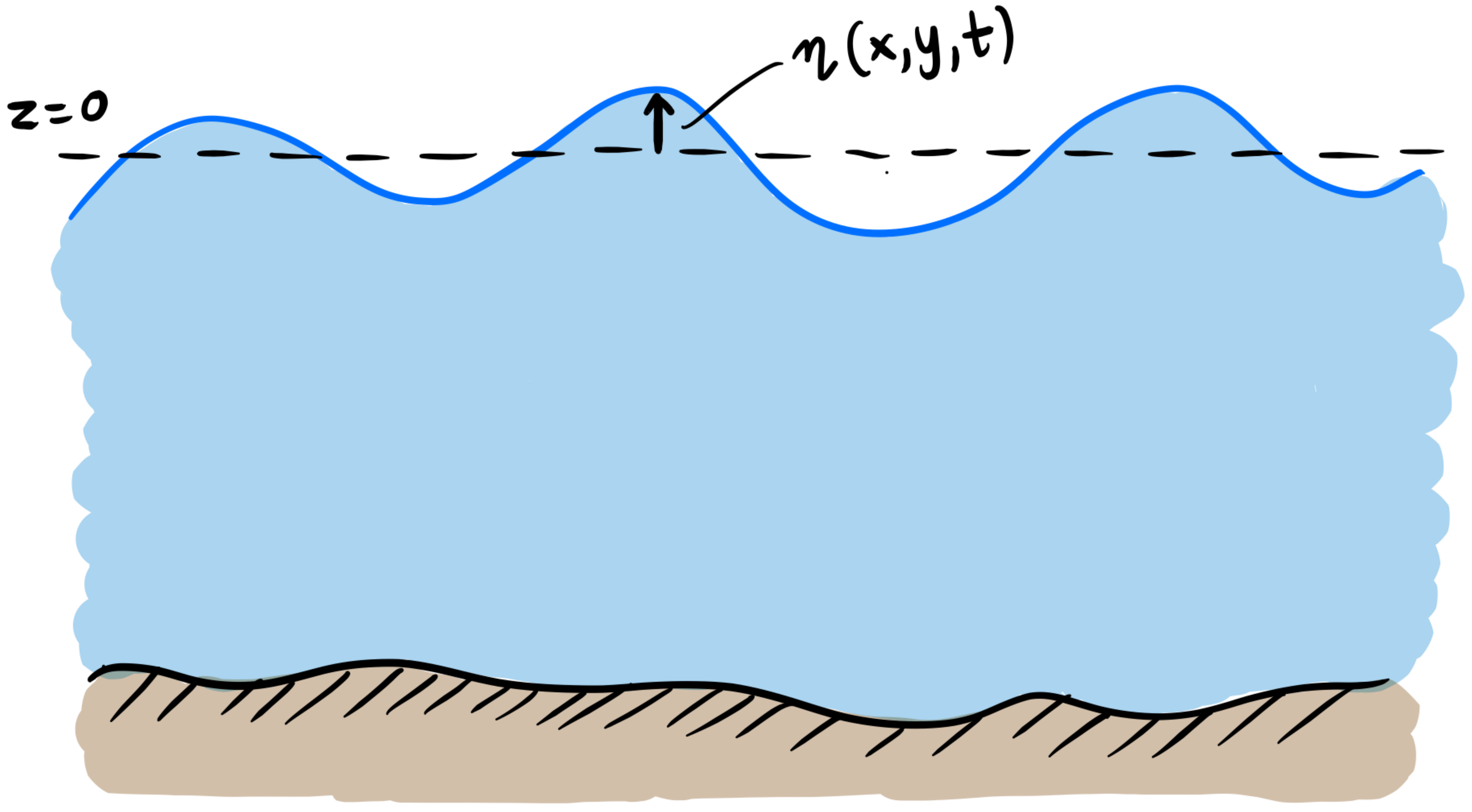
Shallow-water dynamics



vertical momentum equation

$$\underbrace{\frac{\partial w}{\partial t}}_{\text{very small}} + \underbrace{u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}}_{\text{very small}} = \underbrace{-\frac{1}{\varrho_0}\frac{\partial p}{\partial z}}_{\text{LARGE}} \underbrace{- g}_{\text{LARGE}}$$

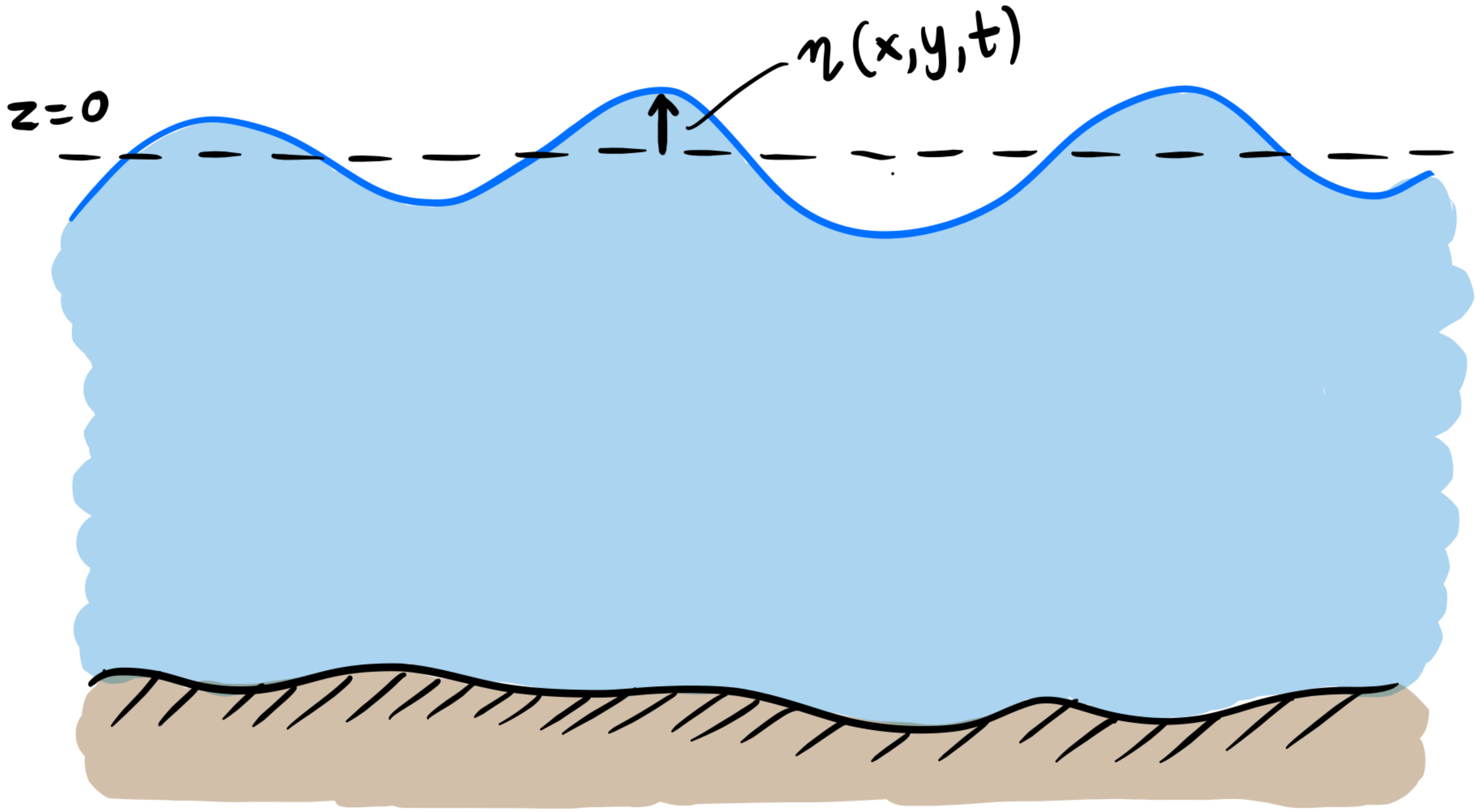
Shallow-water dynamics



vertical momentum equation

$$\underbrace{\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}}_{\text{very small}} = -\underbrace{\frac{1}{\varrho_0}\frac{\partial p}{\partial z}}_{\text{LARGE}} - \underbrace{g}_{\text{LARGE}}$$

Shallow-water dynamics



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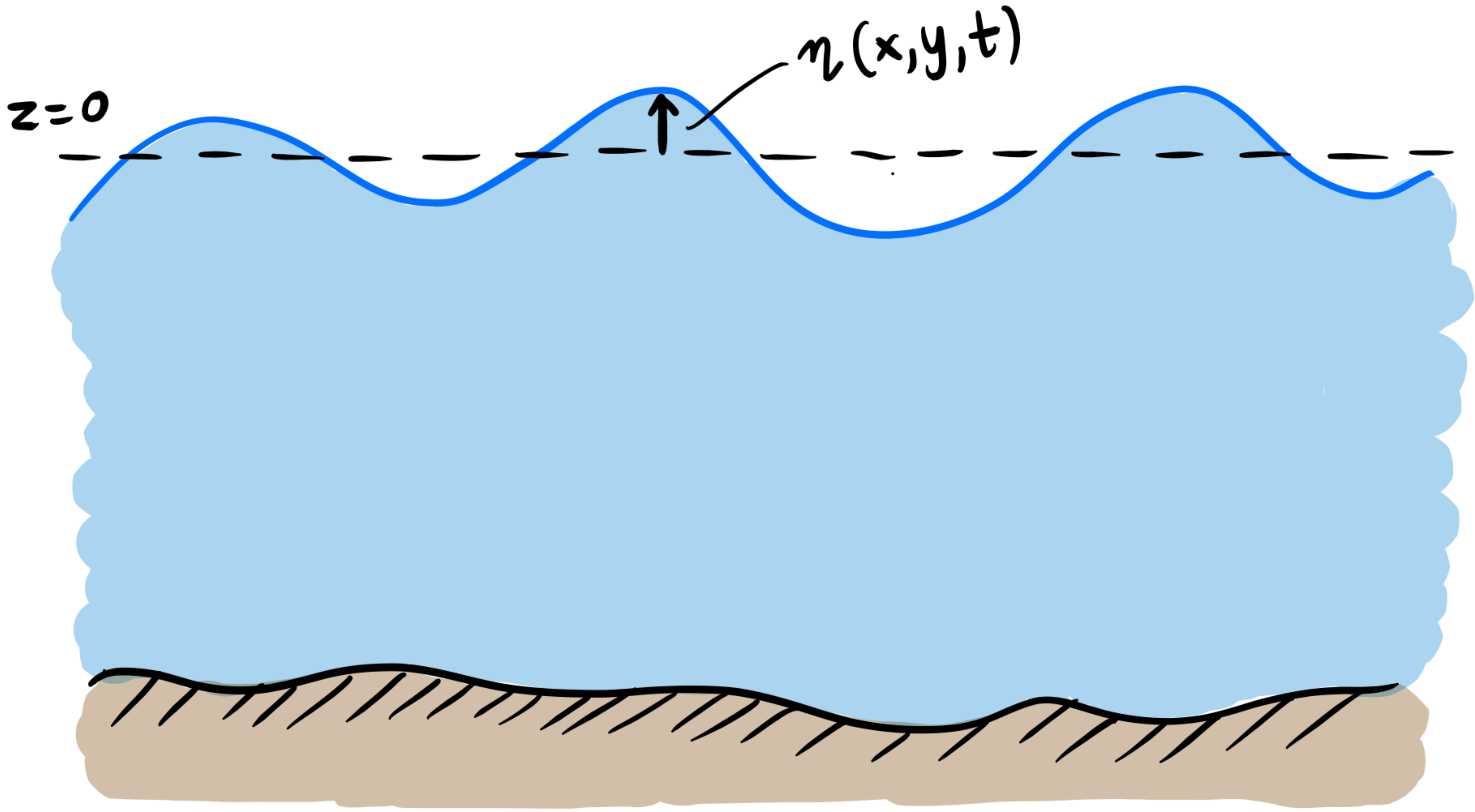
hydrostatic balance

$$\frac{\partial p}{\partial z} = -\varrho_0 g$$

Shallow-water dynamics



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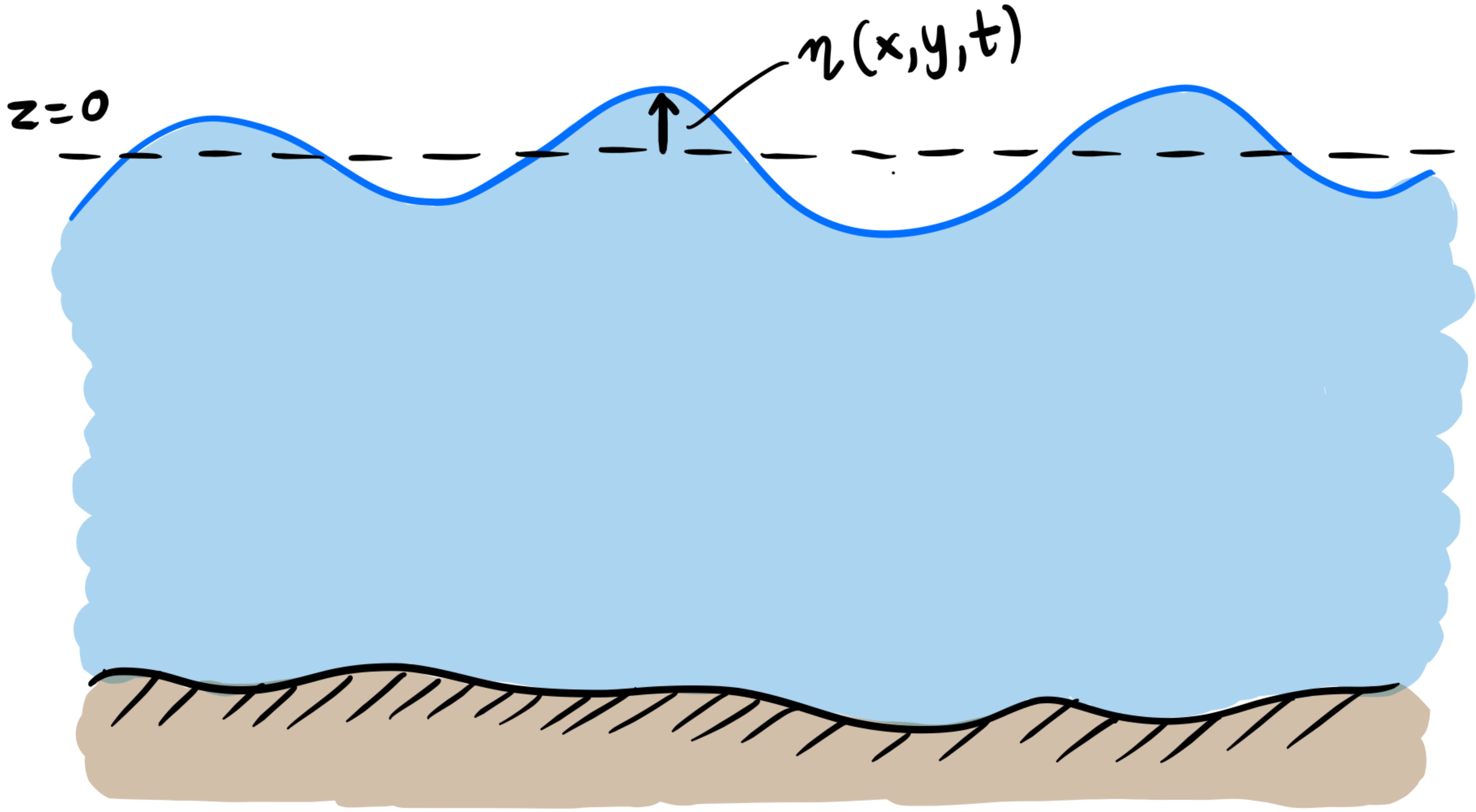
hydrostatic balance

$$\frac{\partial p}{\partial z} = -\varrho_0 g$$

$$p(x, y, z = \eta, t) = 1 \text{ atm}$$

$$\Longrightarrow p(x, y, z, t) = \varrho_0 g(\eta(x, y, t) - z) + 1 \text{ atm}$$

Shallow-water dynamics



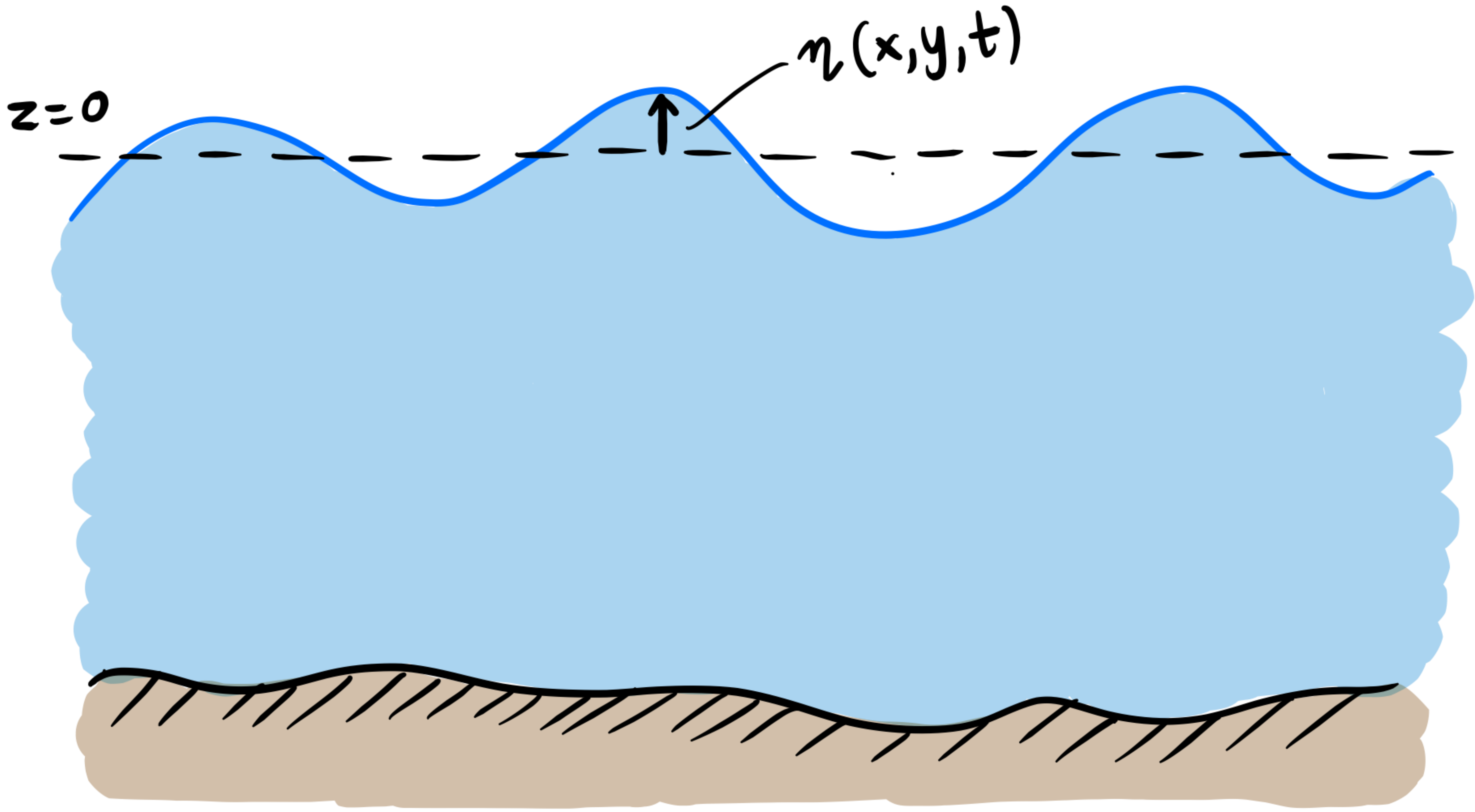
horizontal momentum equations

$$\underbrace{\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}}_{\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}} + f\hat{z} \times \boldsymbol{u} = -\frac{1}{\varrho_0}\nabla p$$

$$\boldsymbol{u} = (u(x,y,t), v(x,y,t))$$

$$\nabla = \frac{\partial}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial}{\partial y}\hat{\boldsymbol{y}}$$

Shallow-water dynamics



horizontal momentum equations

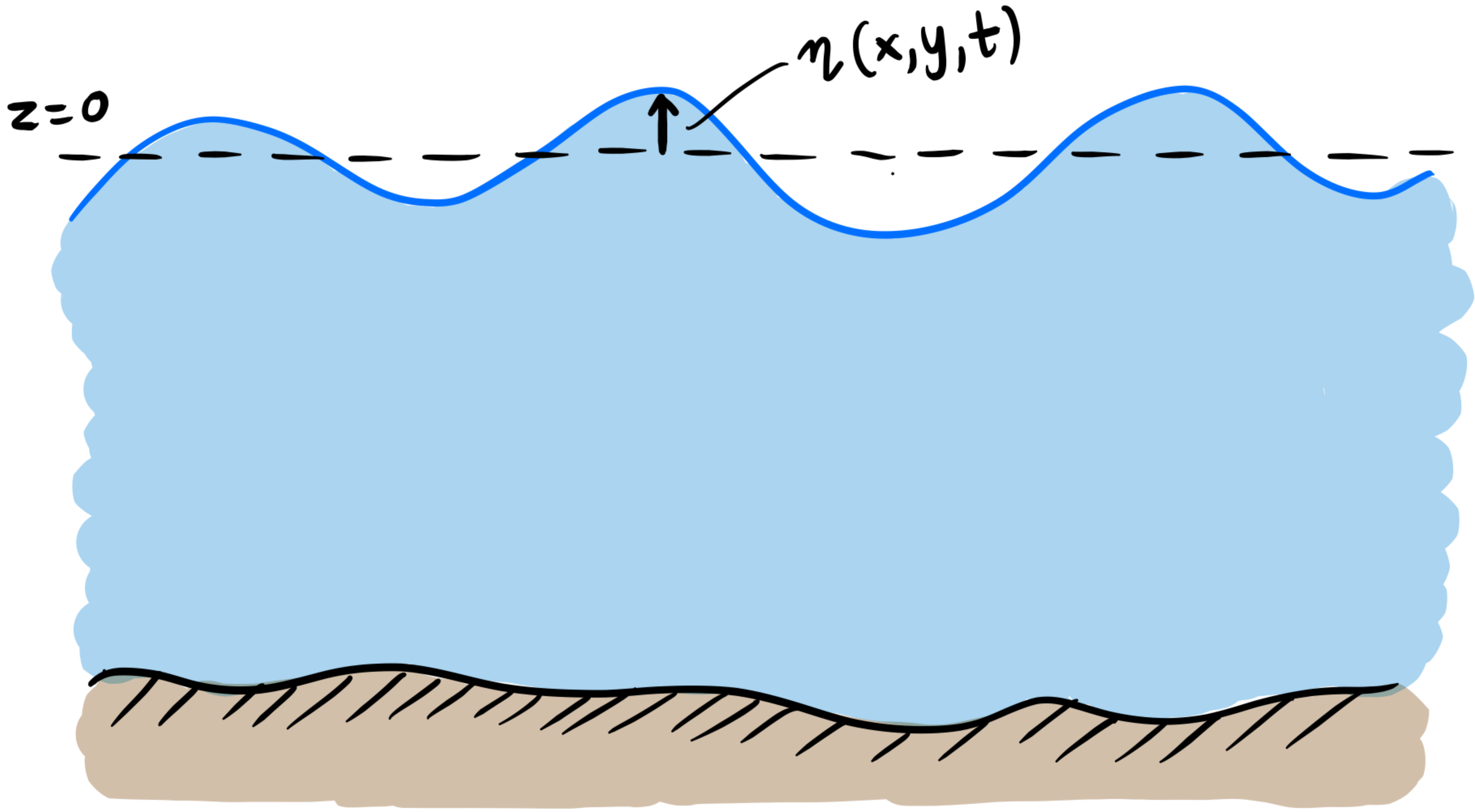
$$\underbrace{\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}}_{\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}} + f\hat{z}\times\boldsymbol{u} = -\underbrace{\frac{1}{\varrho_0}\nabla p}_{\text{use hydrostatic balance}}$$

$$\boldsymbol{u} = (u(x,y,t), v(x,y,t))$$

$$\nabla = \frac{\partial}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial}{\partial y}\hat{\boldsymbol{y}}$$

$$p(x,y,z,t) = \varrho_0 g(\eta(x,y,t) - z) + p_0$$

Shallow-water dynamics



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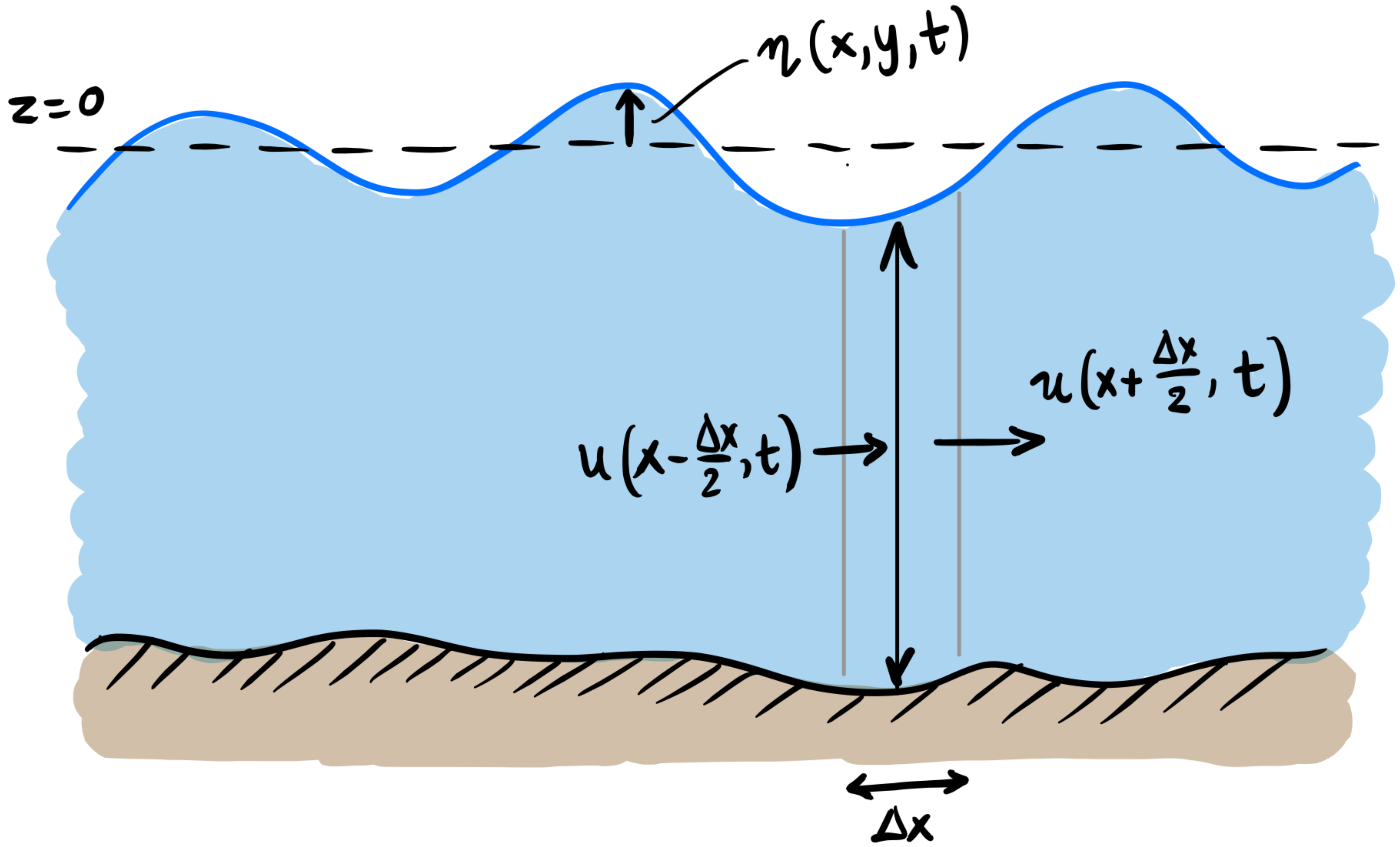
$$p(x,y,z,t) = \varrho_0 g(\eta(x,y,t) - z) + p_0$$

$$\boxed{\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + f\hat{\boldsymbol{z}} \times \boldsymbol{u} = -g\nabla \eta}$$

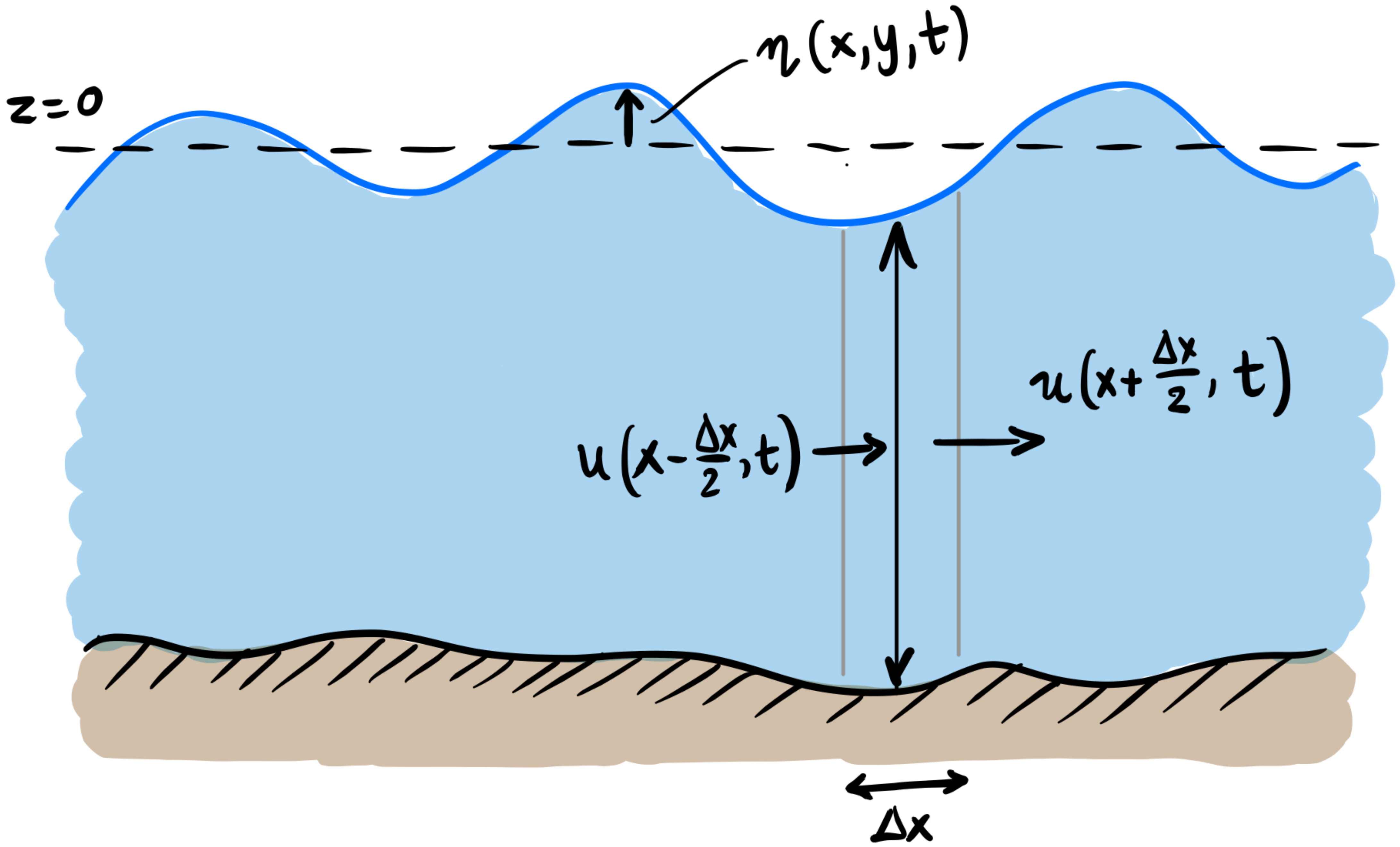
Shallow-water dynamics



mass conservation

constant density $\Longrightarrow$ volume conservation

Shallow-water dynamics

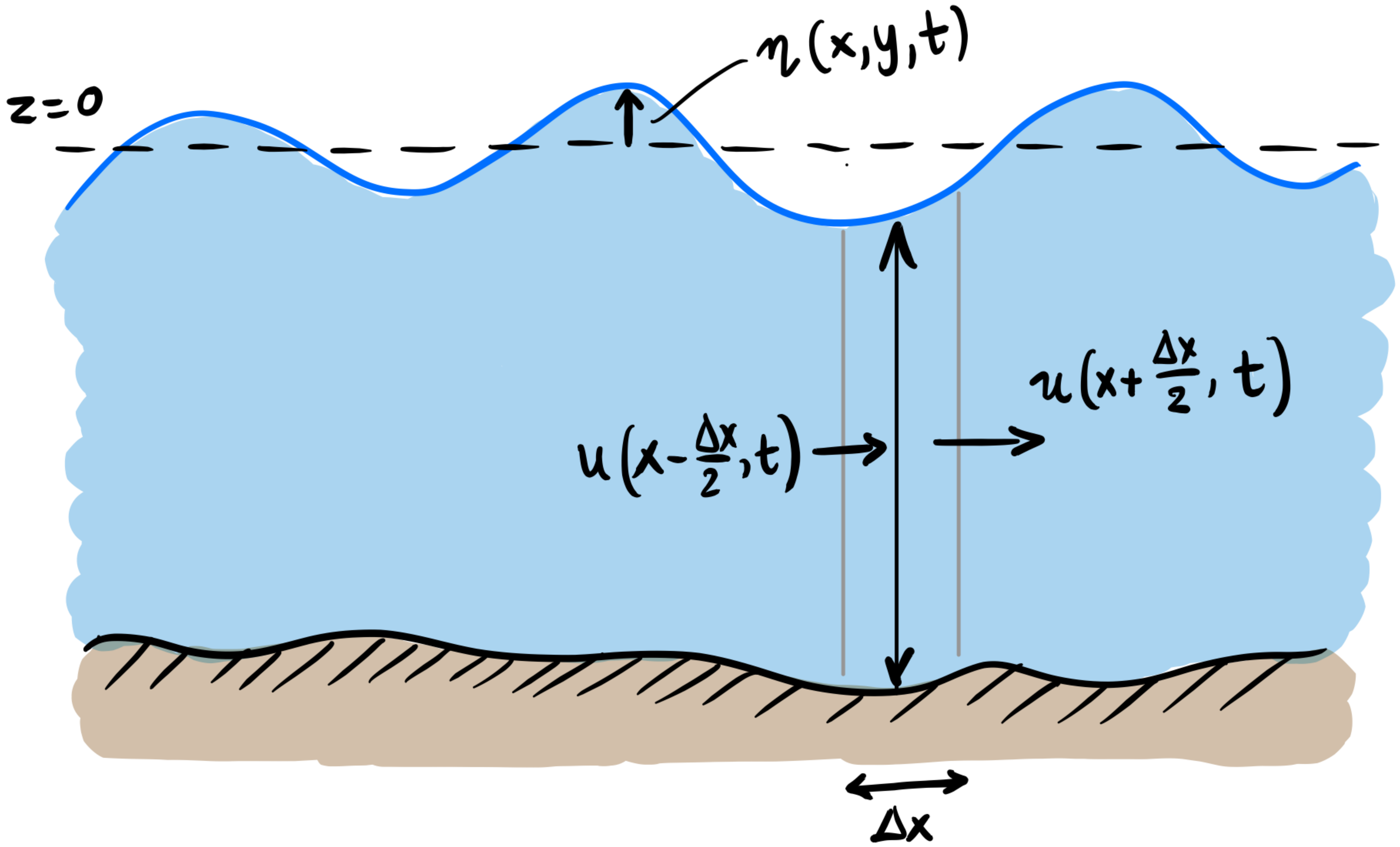


mass conservation

constant density $\Longrightarrow$ volume conservation

volume of  $= h(x,t)\,\Delta x$

Shallow-water dynamics



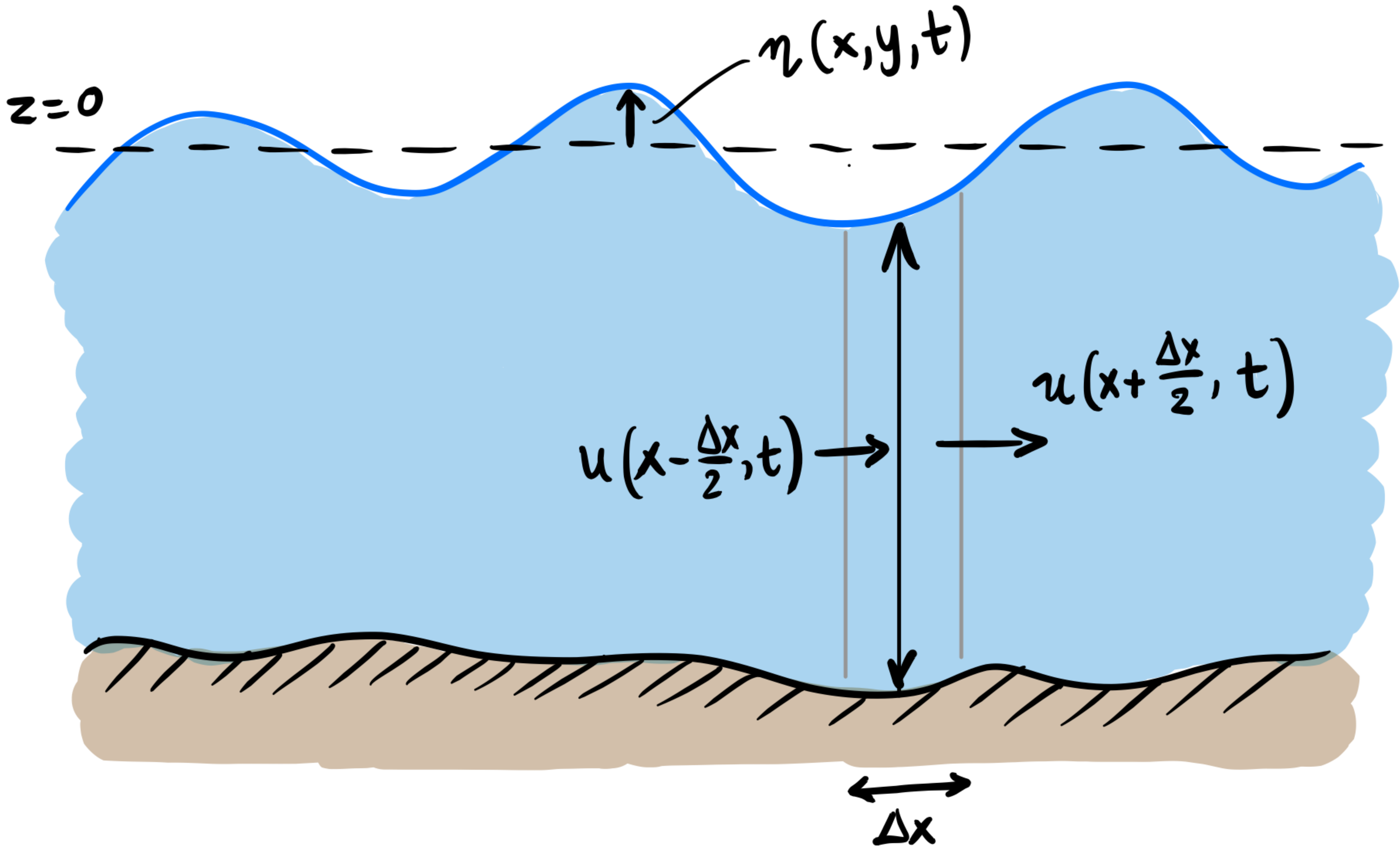
mass conservation

constant density $\Longrightarrow$ volume conservation

volume of $= h(x,t)\ \Delta x$

change of volume $=$ fluid flux into $-$ fluid flux out of

Shallow-water dynamics



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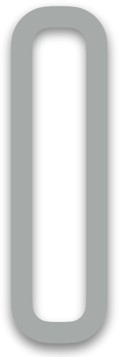
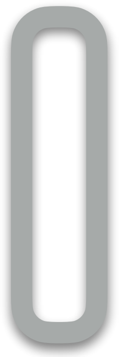
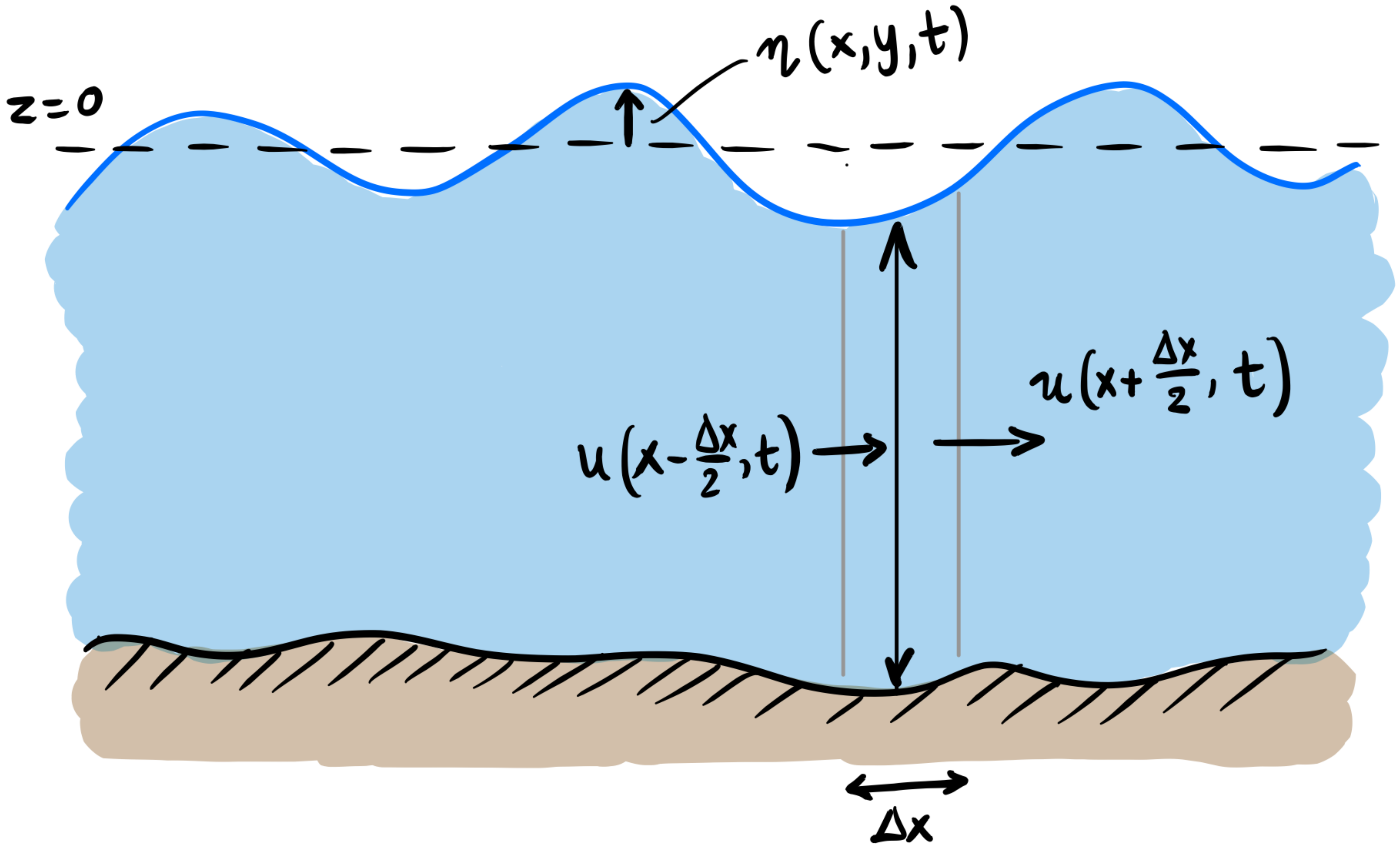
change of volume  $=$ fluid flux into  $-$ fluid flux out of 

$$\Delta x \, \frac{\partial h(x,t)}{\partial t} = u\big(x - \frac{\Delta x}{2}, t\big)\, h\big(x - \frac{\Delta x}{2}, t\big) - u\big(x + \frac{\Delta x}{2}, t\big)\, h\big(x + \frac{\Delta x}{2}, t\big)$$

Shallow-water dynamics



mass conservation

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volume of $=$ $h(x,t)\,\Delta x$

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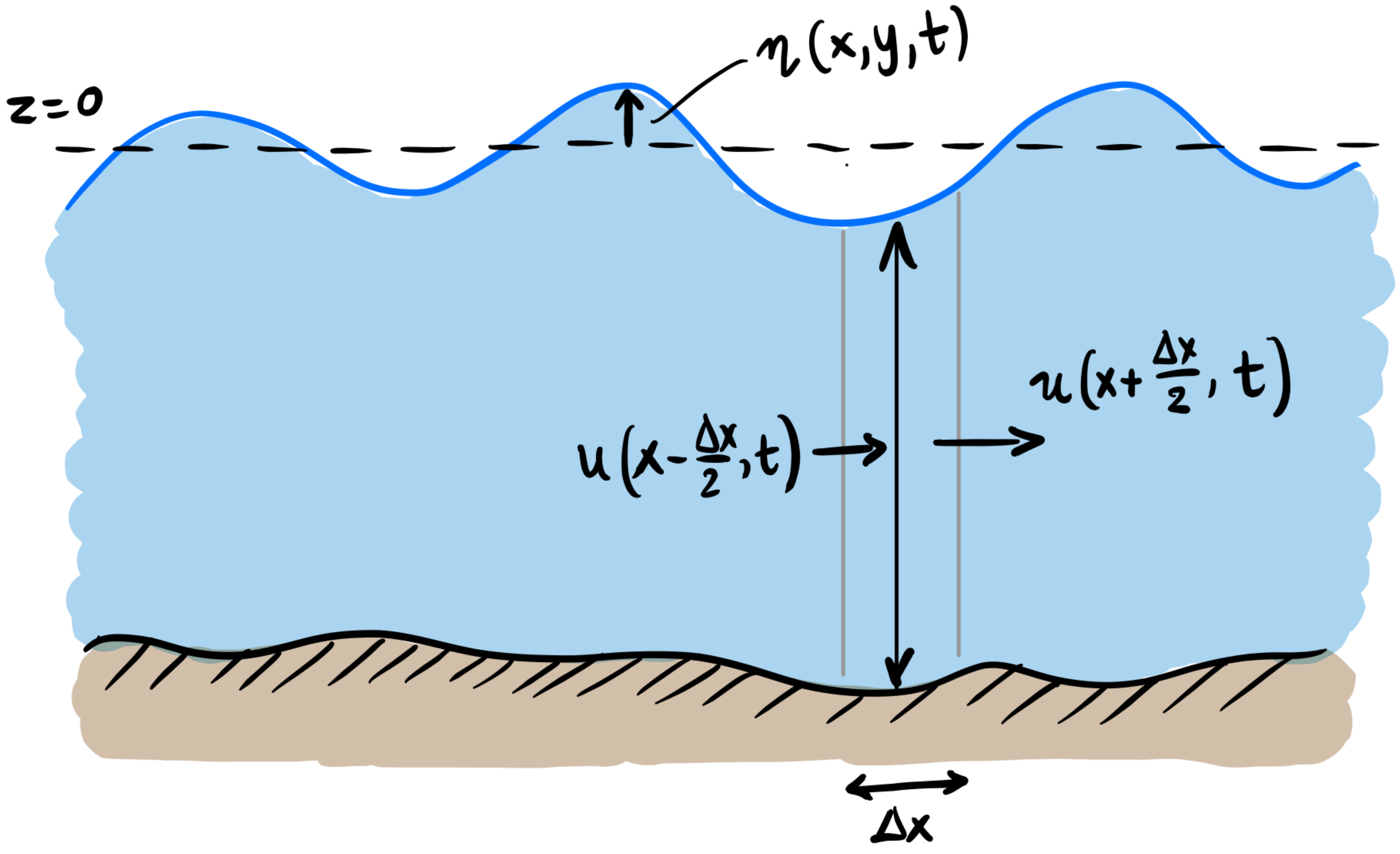
$$= \Delta x\,\frac{u\big(x-\frac{\Delta x}{2},t\big)\,h\big(x-\frac{\Delta x}{2},t\big) - u\big(x+\frac{\Delta x}{2},t\big)\,h\big(x+\frac{\Delta x}{2},t\big)}{\Delta x}$$

Shallow-water dynamics



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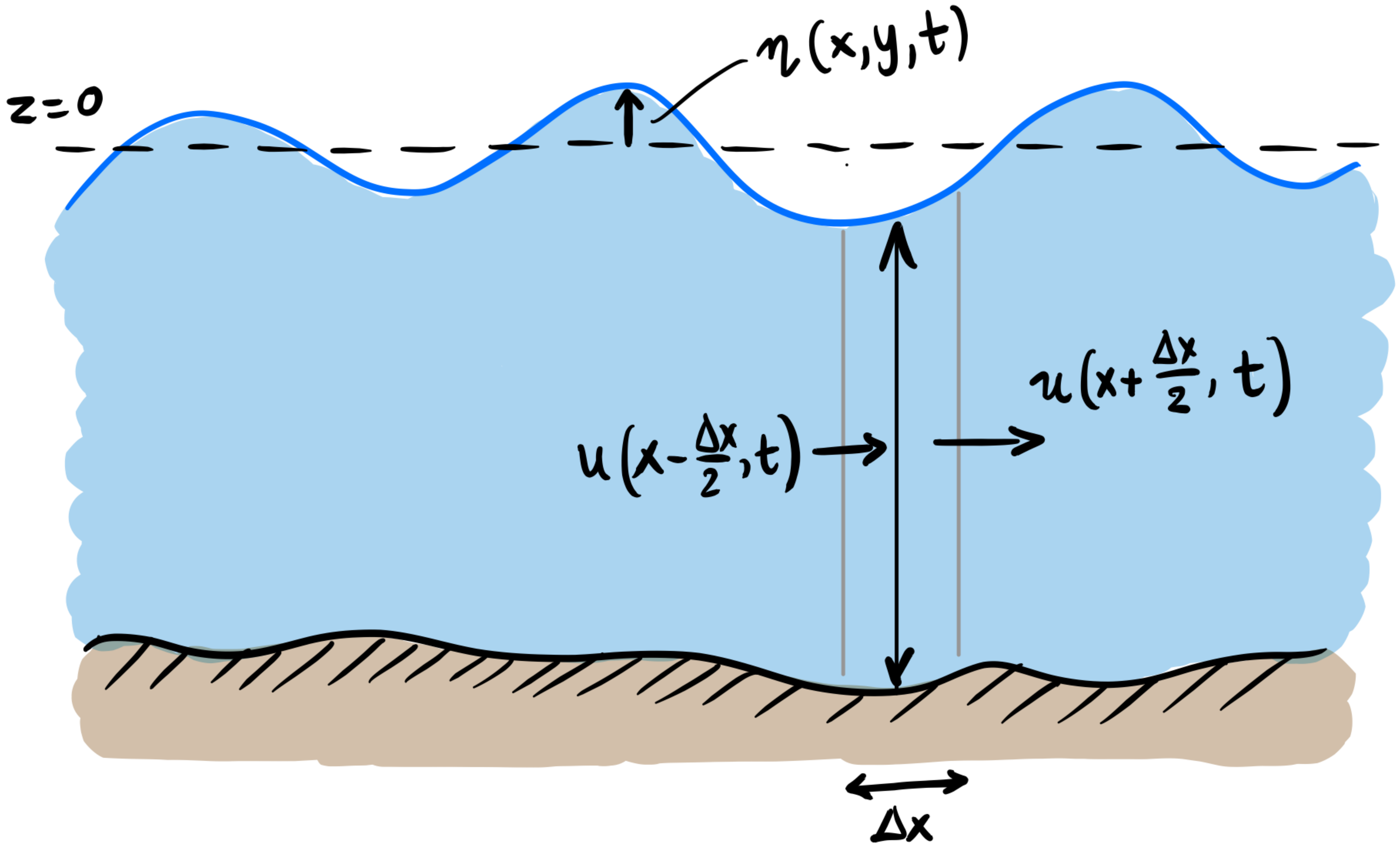
$$\approx -\,\Delta x\,\frac{\partial}{\partial x}\big[h(x,t)u(x,t)\big]$$

Shallow-water dynamics



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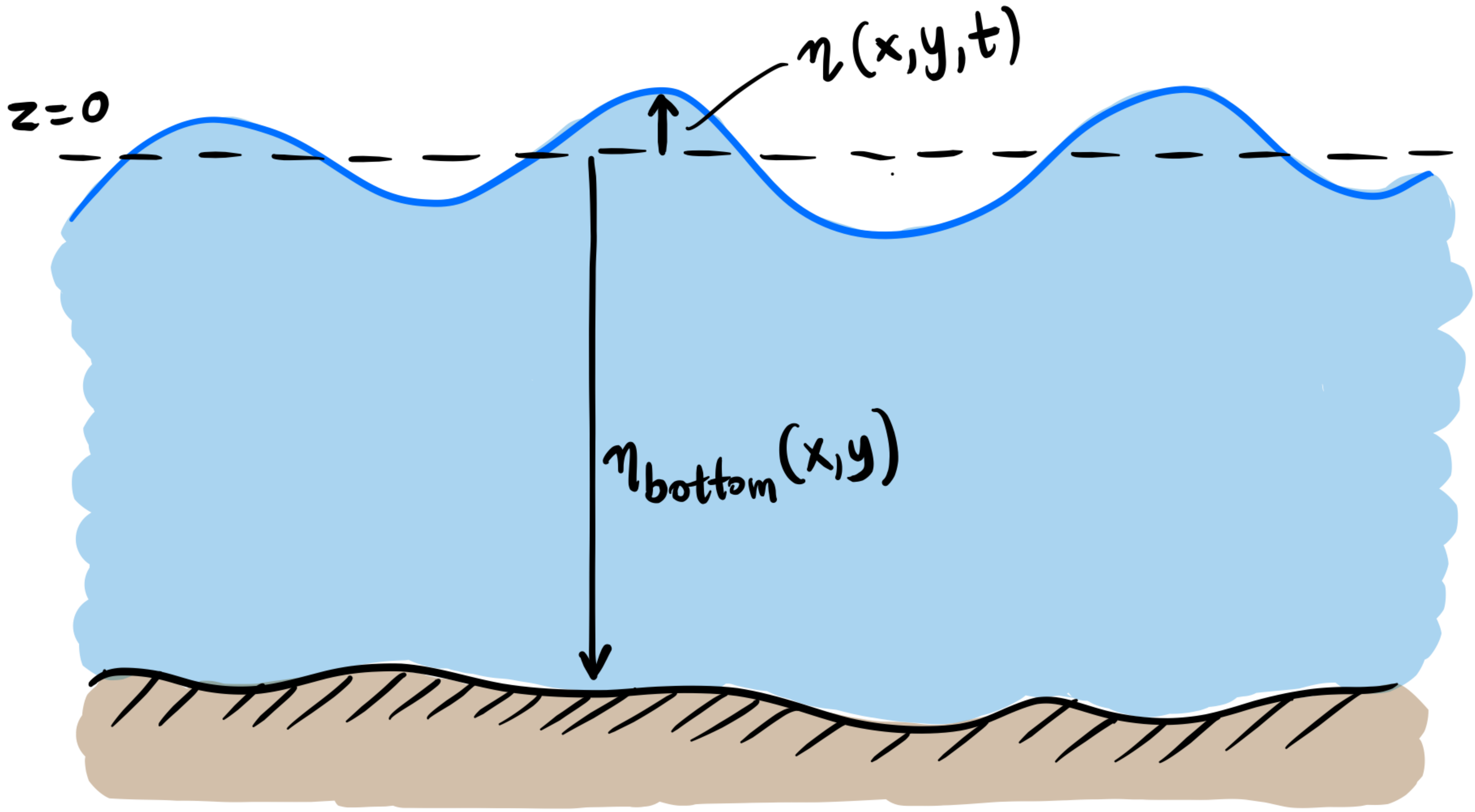
$$\approx -\,\Delta x\frac{\partial}{\partial x}\big[h(x,t)u(x,t)\big]$$

$$\boxed{\frac{\partial h}{\partial t} + \nabla\cdot\big(\boldsymbol{u}h\big) = 0}$$

Shallow-water dynamics



$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + f\hat{\boldsymbol{z}} \times \boldsymbol{u} = -\,g\,\nabla\eta$$

$$\frac{\partial h}{\partial t} + \nabla \cdot \left(\boldsymbol{u}h\right) = 0$$

$$h(x,y,t) = \eta(x,y,t) - \eta_{\text{bottom}}(x,y)$$

$$\boldsymbol{u} = (u(x,y,t), v(x,y,t))$$

$$\nabla = \frac{\partial}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial}{\partial y}\hat{\boldsymbol{y}}$$

admit wave solutions
(waves that “live” on the fluid’s surface)

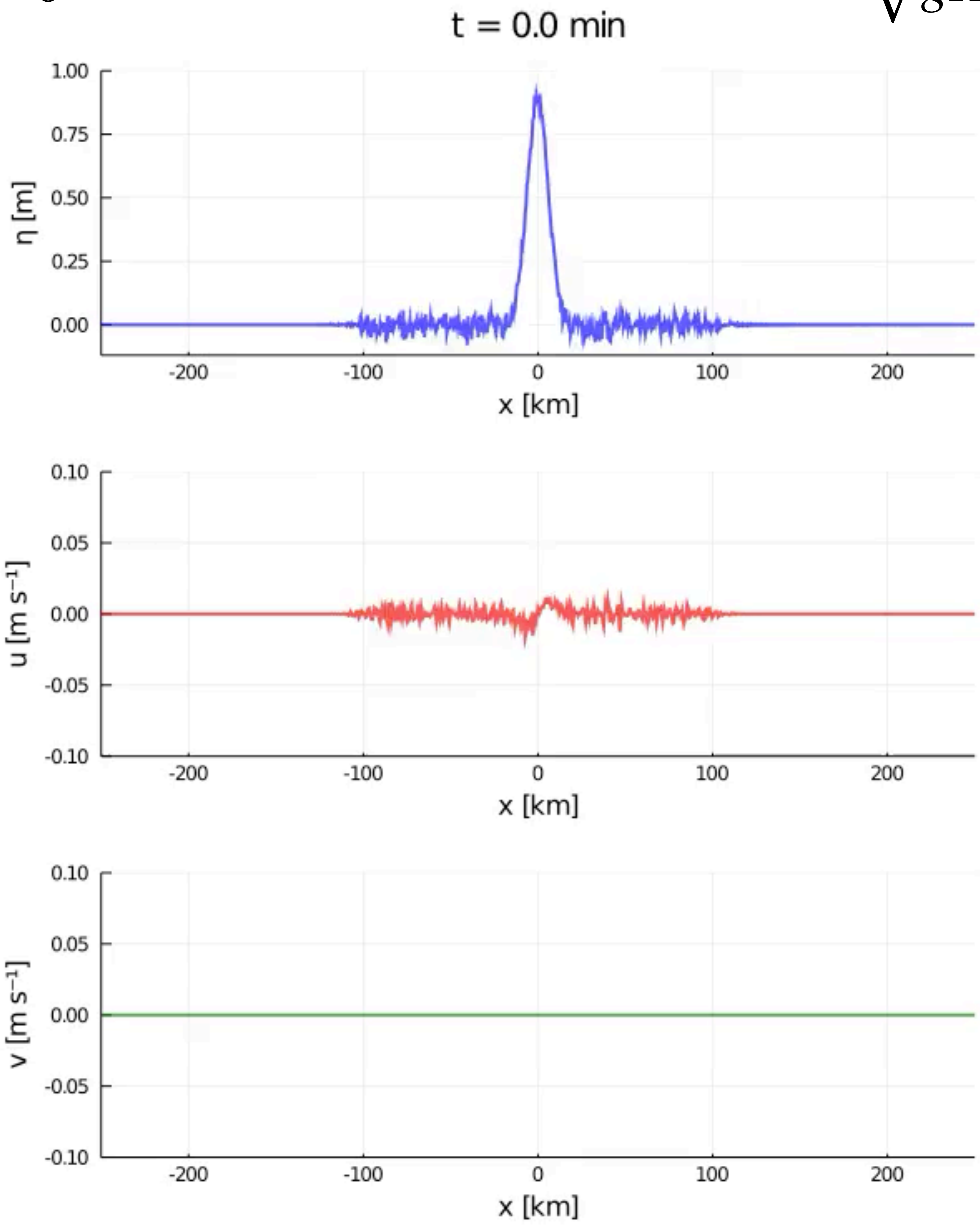
$f = 0$

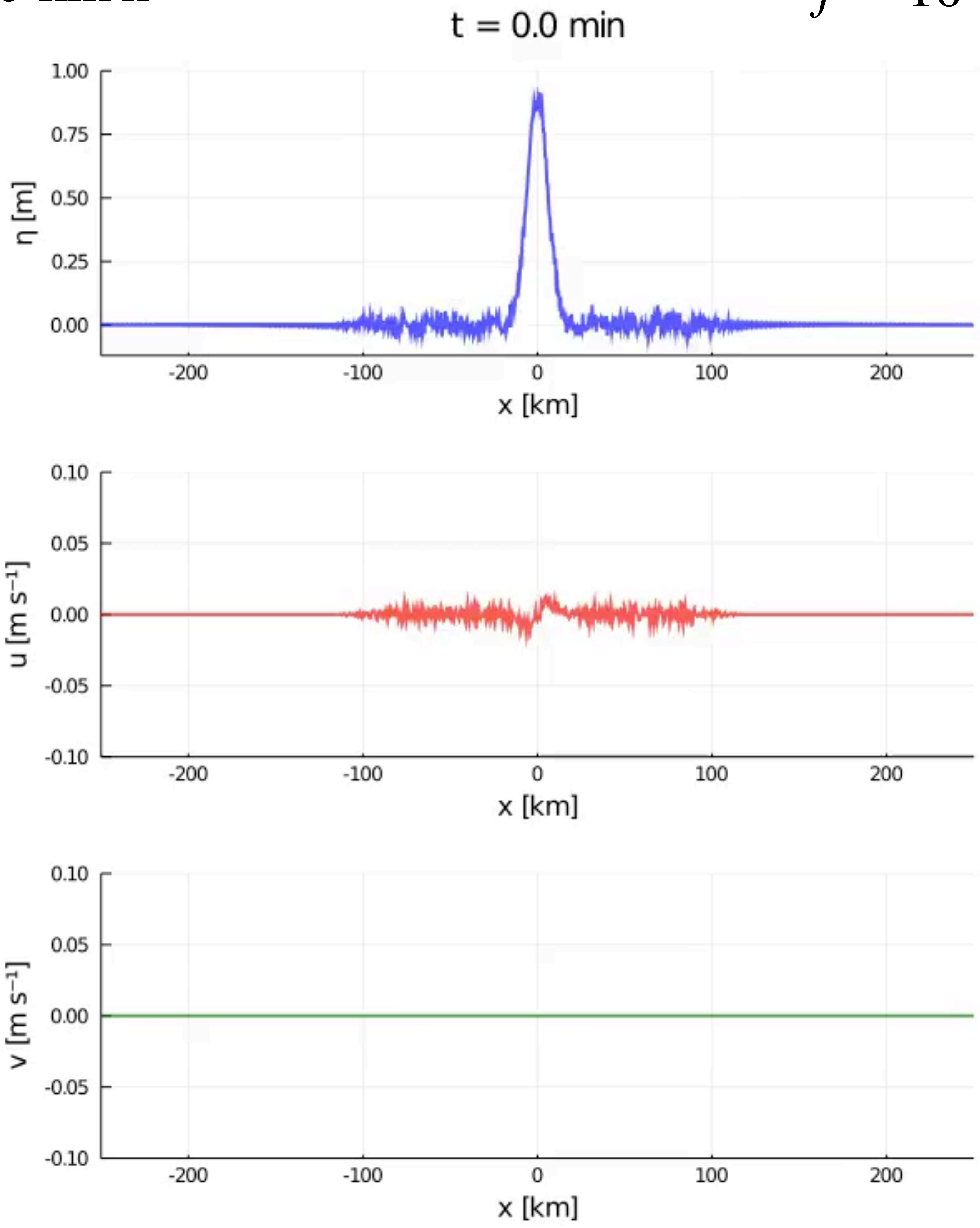
non-rotating

$\sqrt{gH} = 160 \ \mathrm{km\,h^{-1}}$

rotating

$f = 10^{-2} \ \mathrm{s^{-1}}$

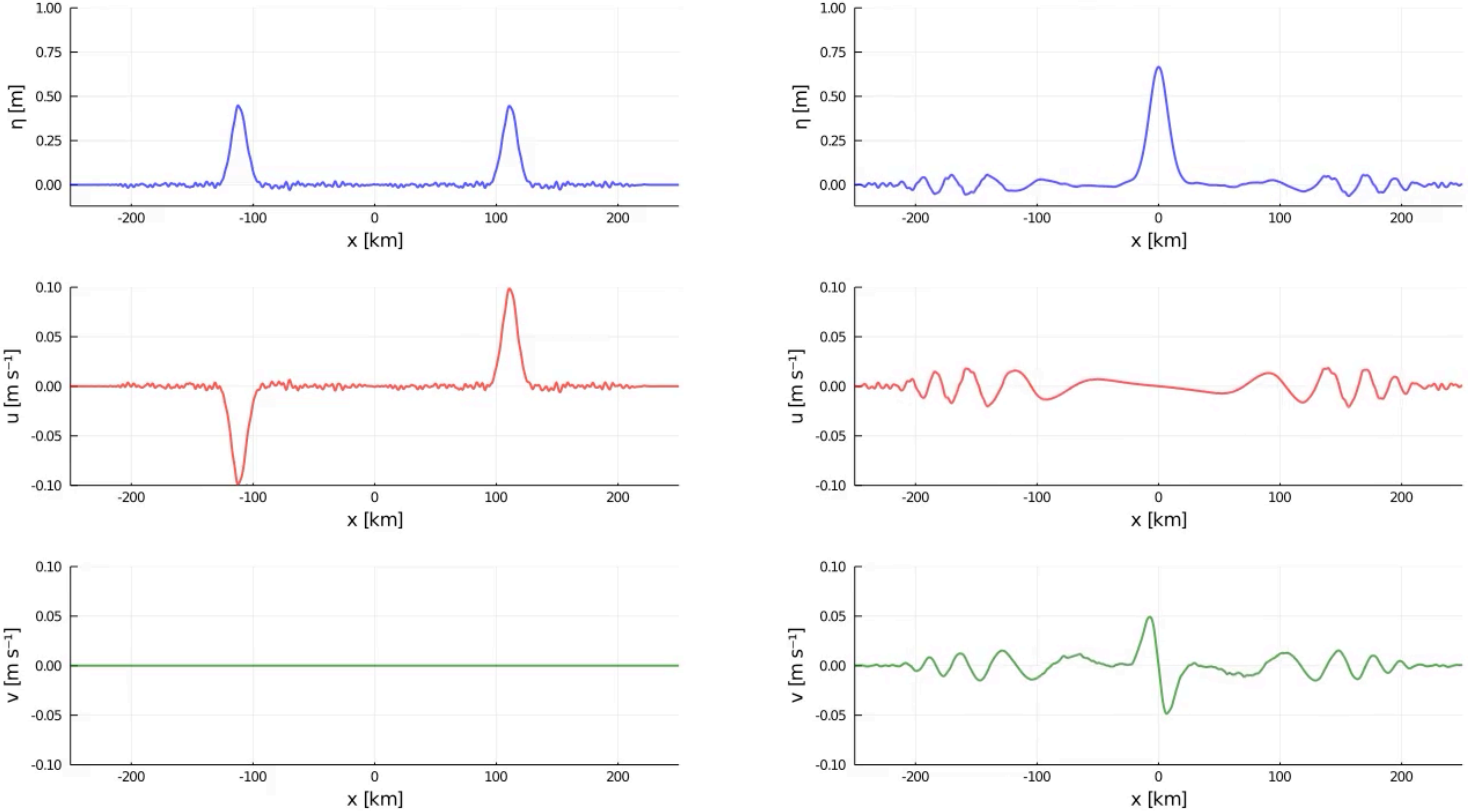




$f = 0$ **non-rotating** $\sqrt{gH} = 160\ \mathrm{km\,h^{-1}}$ **rotating** $f = 10^{-2}\ \mathrm{s^{-1}}$

t = 41.7 min

t = 89.0 min



Rotating shallow-water dynamics

horizontal
momentum eqs

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + f\hat{z} \times \boldsymbol{u} = -g\nabla\eta$$

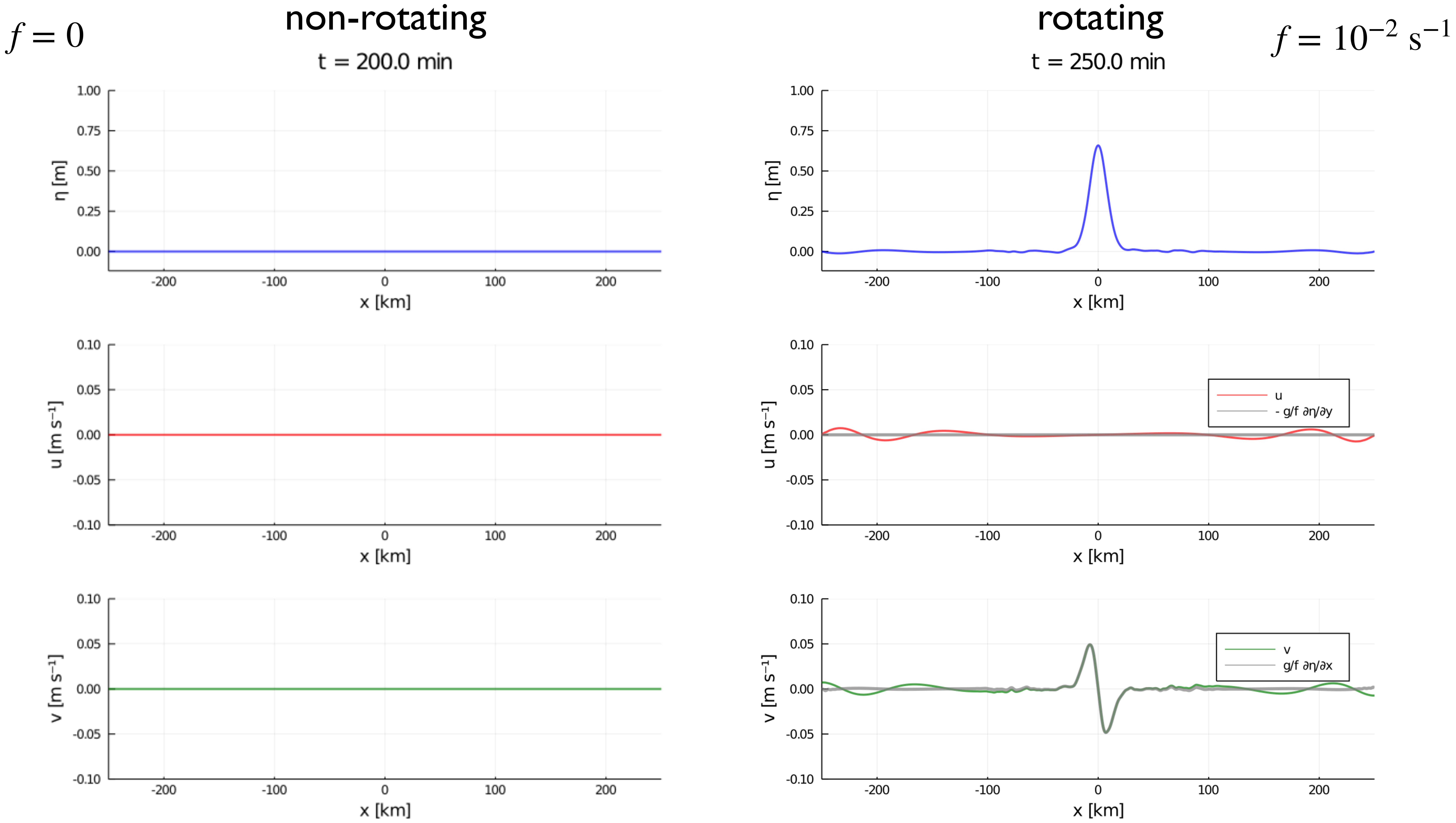
after the dust settles…

$$f\hat{z} \times \boldsymbol{u} \approx -g\nabla\eta$$

Coriolis $\approx$ pressure gradient

geostrophic balance

$$u_{\text{geostrophic}} = -\frac{g}{f}\frac{\partial\eta}{\partial y} \qquad v_{\text{geostrophic}} = +\frac{g}{f}\frac{\partial\eta}{\partial x}$$



$f = 0$
non-rotating
t = 200.0 min
rotating
t = 250.0 min
$f = 10^{-2}\ s^{-1}$
η [m]
u [m s⁻¹]
v [m s⁻¹]
x [km]
u
- g/f ∂η/∂y
v
g/f ∂η/∂x

Rotating shallow-water dynamics

Slow motions approximately in geostrophic balance

~ days (atmos)/weeks (ocean)

"weather"

$$u_{\text{geostrophic}} = -\frac{\partial}{\partial y}\left(\frac{p}{\varrho f}\right)$$

$$v_{\text{geostrophic}} = +\frac{\partial}{\partial x}\left(\frac{p}{\varrho f}\right)$$

Fast-travelling motions

~ hours

"noise"

Flows in Geostrophic Balance

$$u_{\text{geostrophic}} = -\frac{\partial}{\partial y}\left(\frac{p}{\varrho f}\right) \qquad v_{\text{geostrophic}} = +\frac{\partial}{\partial x}\left(\frac{p}{\varrho f}\right)$$

Evolve much slower than gravity waves

Flows in Geostrophic Balance

$$u_{\text{geostrophic}} = -\frac{\partial}{\partial y}\left(\frac{p}{\varrho f}\right) \qquad v_{\text{geostrophic}} = +\frac{\partial}{\partial x}\left(\frac{p}{\varrho f}\right)$$

Evolve much slower than gravity waves

Incompressible: $$\frac{\partial}{\partial x}u_{\text{geostrophic}} + \frac{\partial}{\partial y}v_{\text{geostrophic}} = 0$$

Flows in Geostrophic Balance

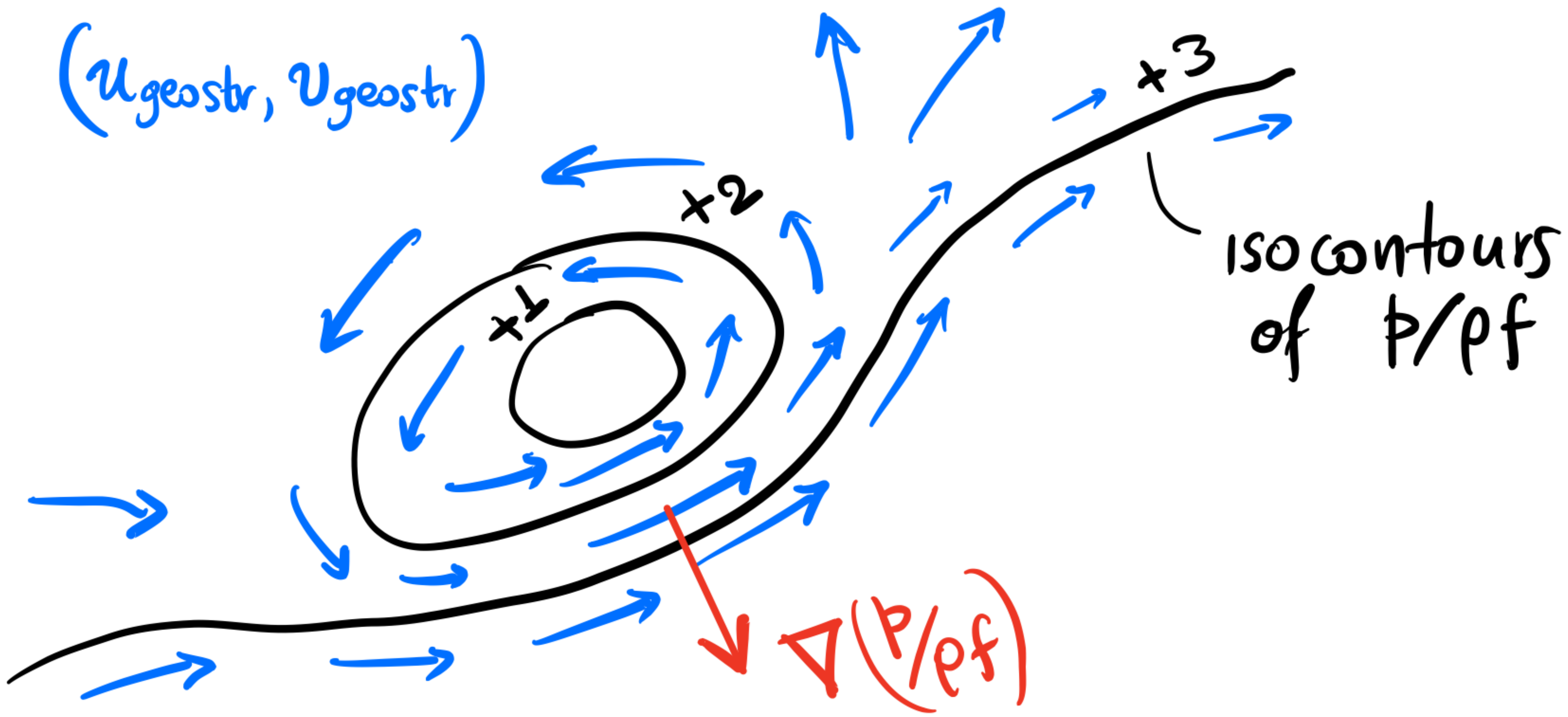
$$u_{\text{geostrophic}} = -\frac{\partial}{\partial y}\left(\frac{p}{\varrho f}\right) \qquad v_{\text{geostrophic}} = +\frac{\partial}{\partial x}\left(\frac{p}{\varrho f}\right)$$

Evolve much slower than gravity waves

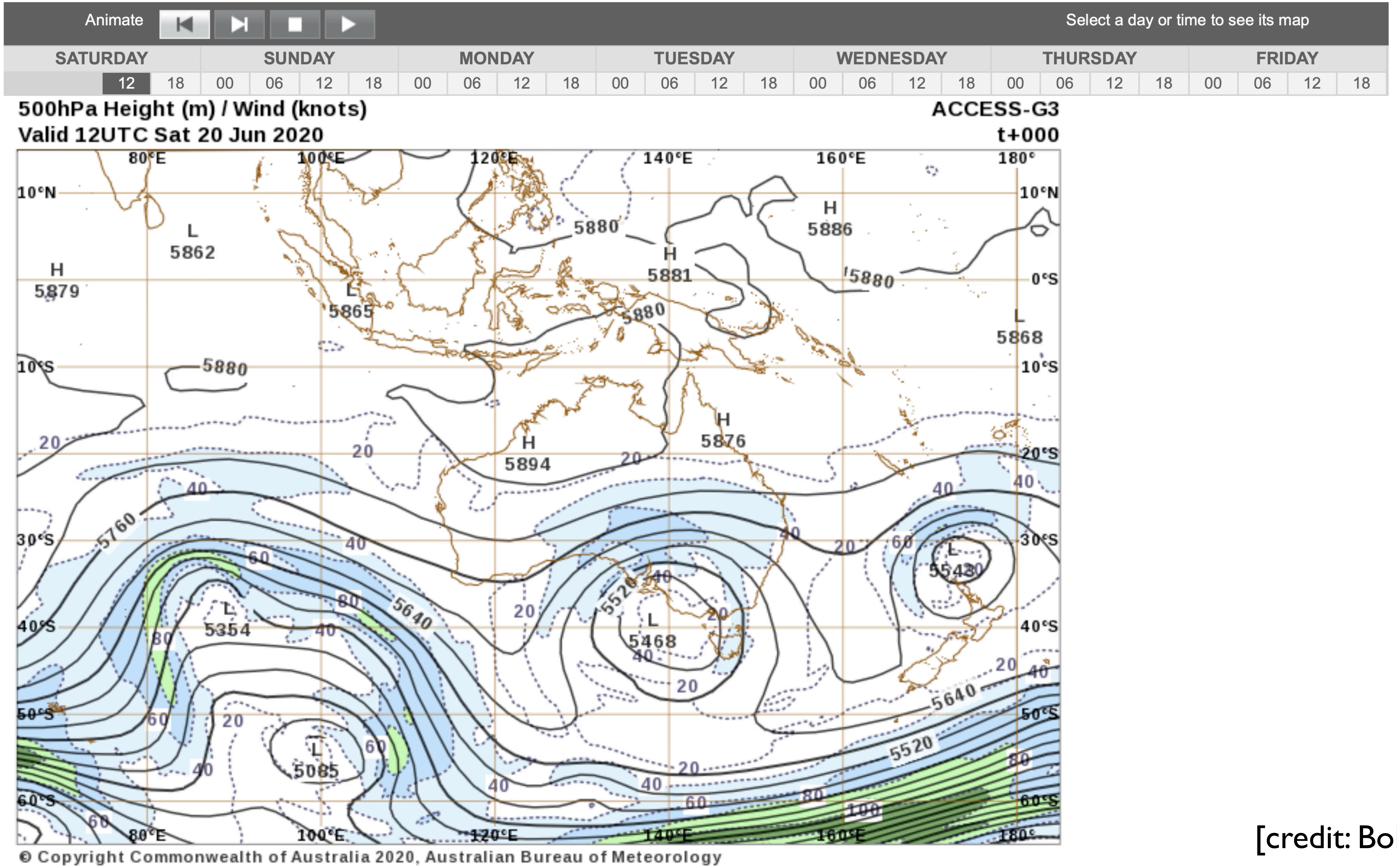
Incompressible: $$\frac{\partial}{\partial x}u_{\text{geostrophic}} + \frac{\partial}{\partial y}v_{\text{geostrophic}} = 0$$

Flow follows contours of constant $p/\varrho f$

$$\nabla\left(\frac{p}{\varrho f}\right) \cdot \boldsymbol{u}_{\text{geostrophic}} = 0$$



Weather maps are all about **Q**uasi-**G**eostrophy



[credit: BoM]

What if *we don't care* about "noise" (=gravity waves)

and *we just want to know*

about the "weather" (almost geostrophically balanced flow)?

Rotating shallow-water dynamics

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \\ \eta \end{pmatrix} = \cdots$$

filter out
fast gravity waves

Quasi-Geostrophic dynamics

$$\frac{\partial}{\partial t} p = \cdots$$

One variable suffices to obtain the flow

Let’s change pace.

How does QG dynamics relates
to 2D turbulence?

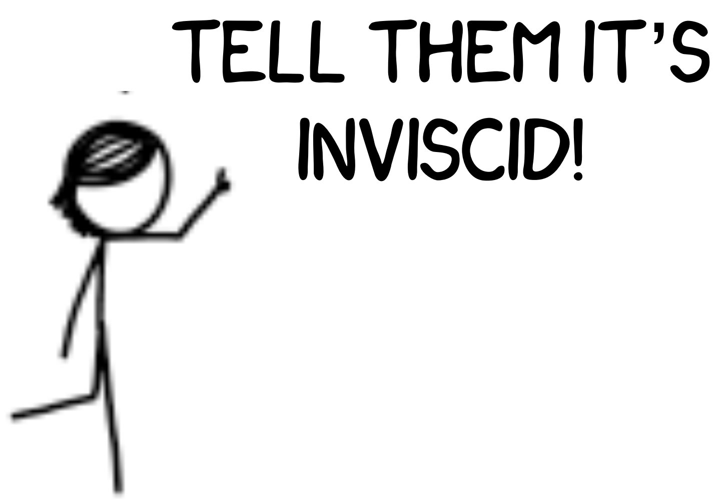
(Incompressible 2D flow = Quasi-Geostrophy **without Earth’s curvature**)

Incompressible 2D flow

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\varrho_0} \nabla p \qquad \boldsymbol{u} = (u(x, y, t), v(x, y, t)) \qquad p(x, y, t)$$

$$\nabla \cdot \boldsymbol{u} = 0$$

Incompressible 2D flow



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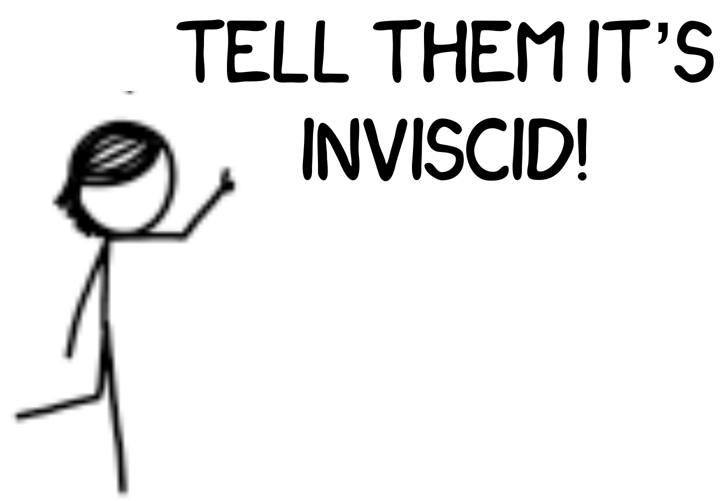
$$p(x, y, t)$$

$$\nabla \cdot \boldsymbol{u} = 0$$

Incompressible 2D flow



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$$\nabla \cdot \boldsymbol{u} = 0$$

$$(u, v) = (-\partial_y \psi, \partial_x \psi)$$

Incompressible 2D flow

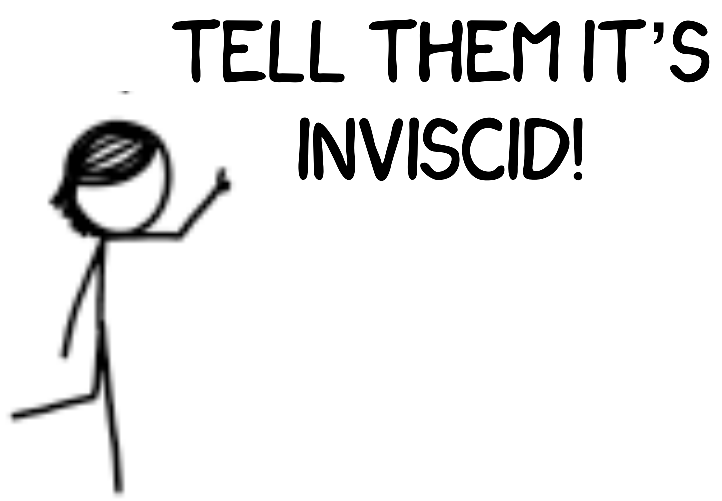
$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\varrho_0} \nabla p \qquad \boldsymbol{u} = (u(x, y, t), v(x, y, t)) \qquad p(x, y, t)$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$(u, v) = (-\partial_y \psi, \partial_x \psi)$$

vorticity $\quad (\nabla \times \boldsymbol{u}) \cdot \hat{z} = \nabla^2 \psi$

Incompressible 2D flow



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$$\nabla \cdot \boldsymbol{u} = 0$$

take the curl $\nabla \times$

$$(u, v) = (-\partial_y \psi, \partial_x \psi)$$

vorticity $\quad (\nabla \times \boldsymbol{u}) \cdot \hat{z} = \nabla^2 \psi$

incompressible 2D flow

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla \right) \nabla^2 \psi = 0$$

$$(u, v) = (-\partial_y \psi, \partial_x \psi)$$

Incompressible 2D flow = Quasi-Geostrophy **on f-plane**

Incompressible 2D flow

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\nabla^2\psi = 0$$

$$(u, v) = (-\partial_y\psi, \partial_x\psi)$$

QG **on f-plane**

(i.e., without Earth's curvature)

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\nabla^2\left(\frac{p}{\varrho_0 f}\right) = 0$$

$$(u, v) = \left(\; -\frac{\partial}{\partial y}\left(\frac{p}{\varrho_0 f}\right), \frac{\partial}{\partial x}\left(\frac{p}{\varrho_0 f}\right)\;\right)$$

Note similarity with
passive tracer equation

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\theta = 0$$

***f*-plane**

$$f = f_0 = \text{const}\,.$$

(Flat Earth)

β-plane

$$f = f_0 + \beta y$$

(Spherical Earth)

[Btw you may find this entertaining: Can the earth be flat ? A physical oceanographer's perspective, arXiv:2001.01521]

Incompressible 2D flow = Quasi-Geostrophy **on f-plane**

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\nabla^2\psi = 0$$

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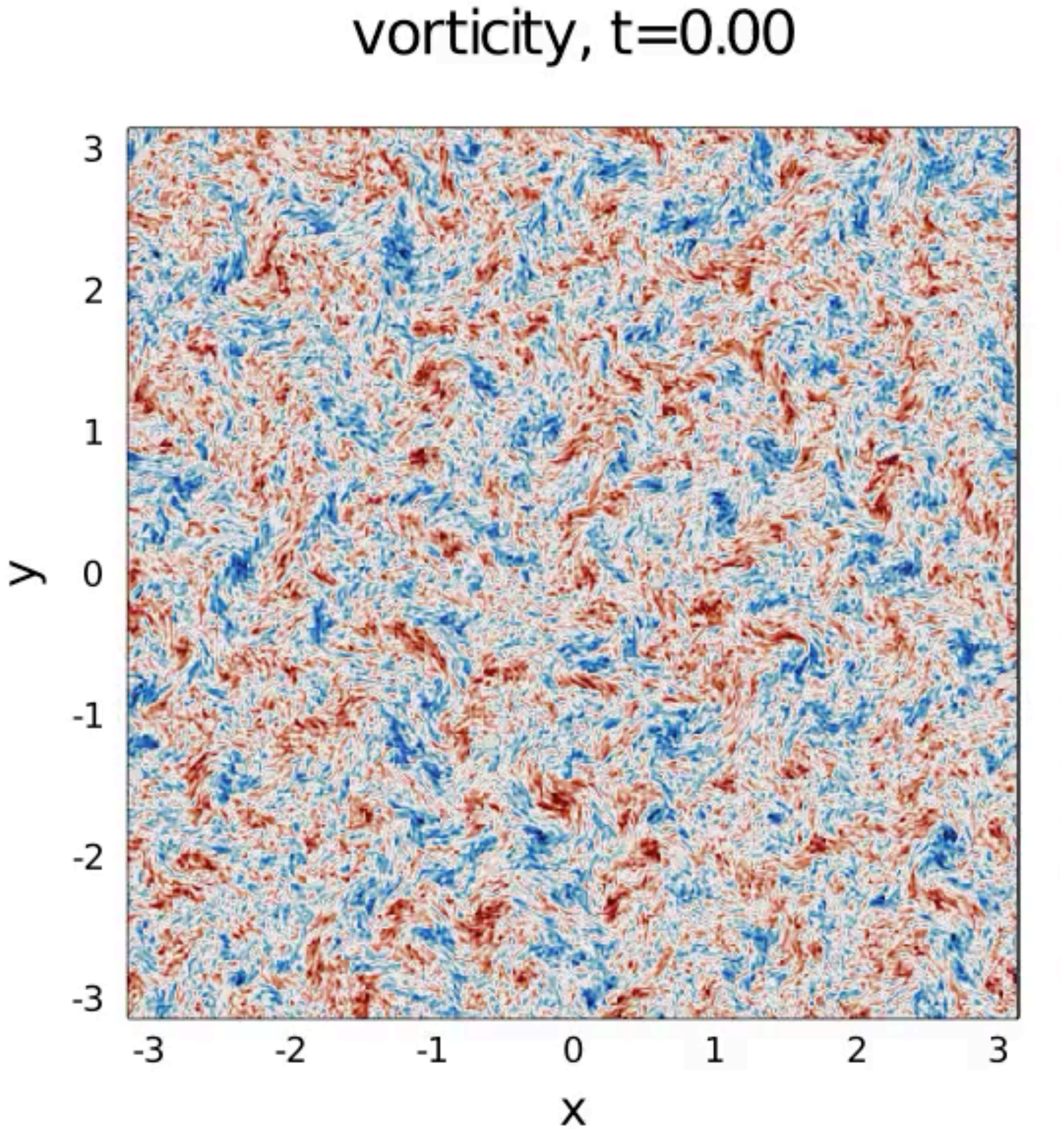
$$(\nabla\times\boldsymbol{u})\cdot\hat{z} = \nabla^2\psi$$



[simulation using GeophysicFlows.jl]

Incompressible 2D flow = Quasi-Geostrophy **on f-plane**

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla\right) \nabla^2 \psi = 0$$

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Note similarity with
passive tracer equation

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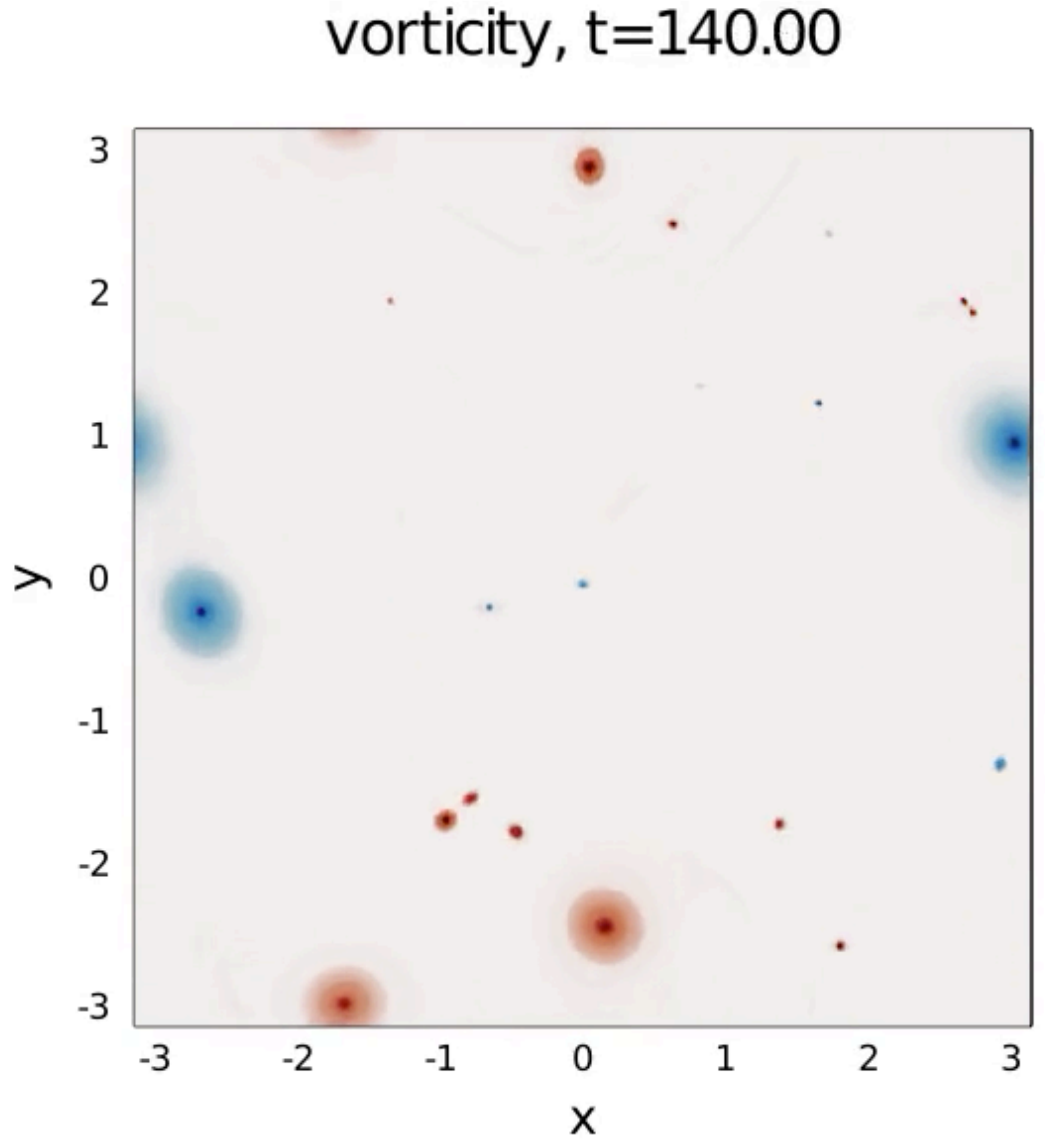
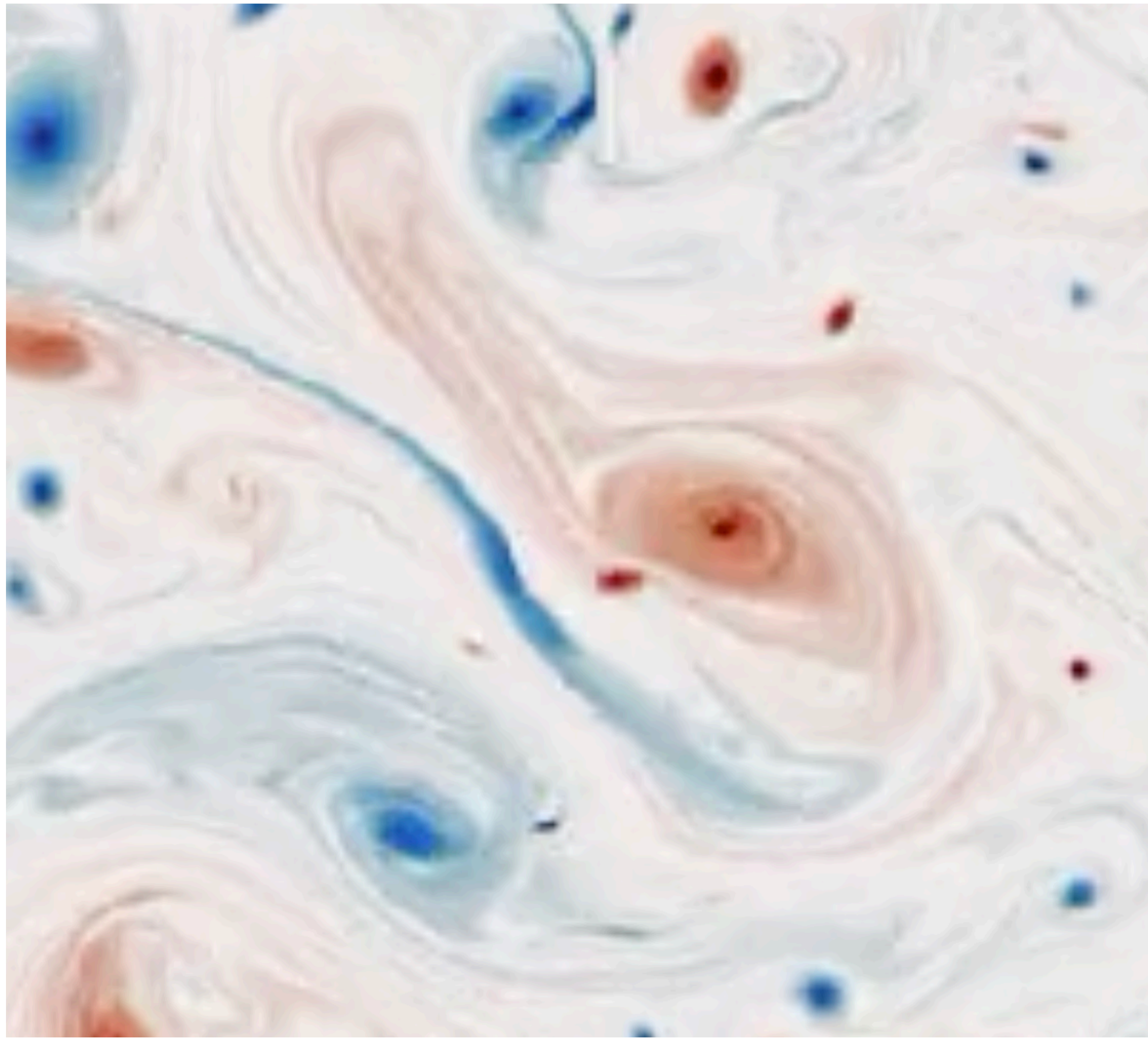


[simulation using GeophysicFlows.jl]

Rotating 3D fluids *resemble* 2D turbulence

2D turbulence **without rotation**



[simulation using GeophysicFlows.jl]

3D fluid in **rotating** tank

[MIT Weather in Tank]

Quasi-Geostrophy **with Earth's curvarture (β-plane)**

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla\right) \underbrace{\left(\nabla^2 \psi + f\right)}_{\text{PV}} = 0$$

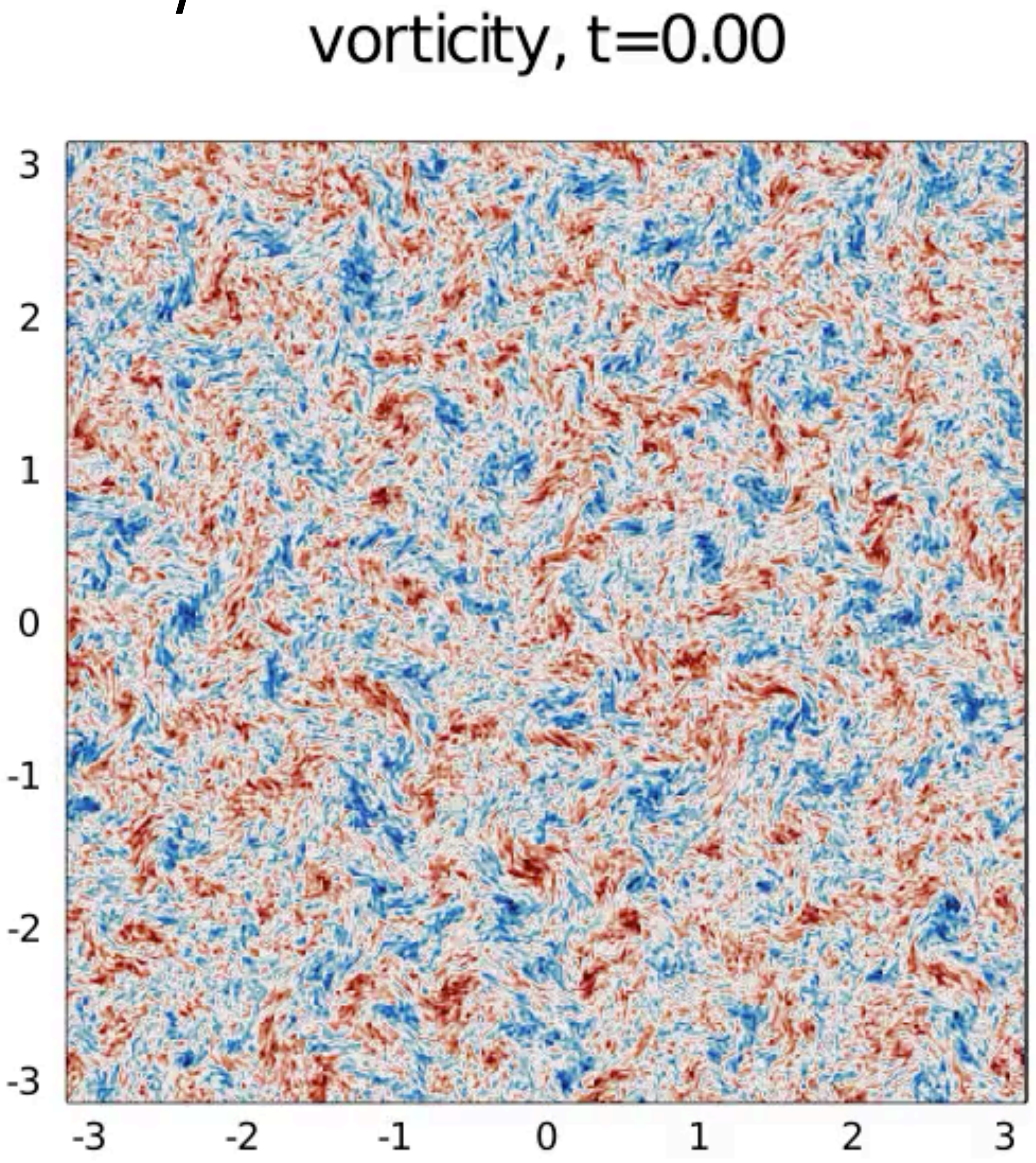
$$(u, v) = (-\partial_y \psi, \partial_x \psi) \ , \qquad \psi = \frac{p}{\varrho_0 f}$$

What's materially conserved is the Potential Vorticity (PV)

Quasi-Geostrophy **with Earth’s curvature (β-plane)**

non-rotating

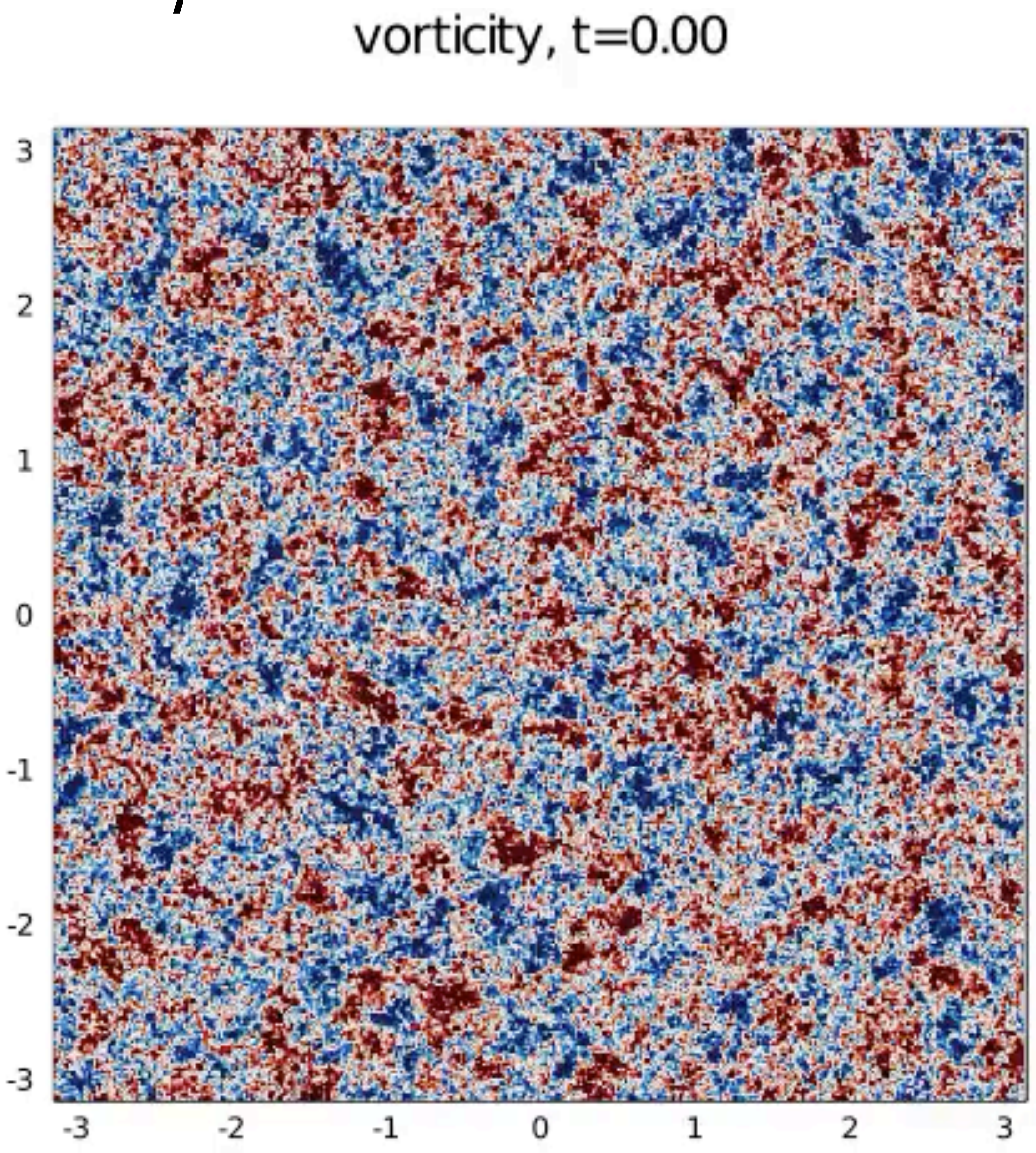
$\nabla^2\psi$



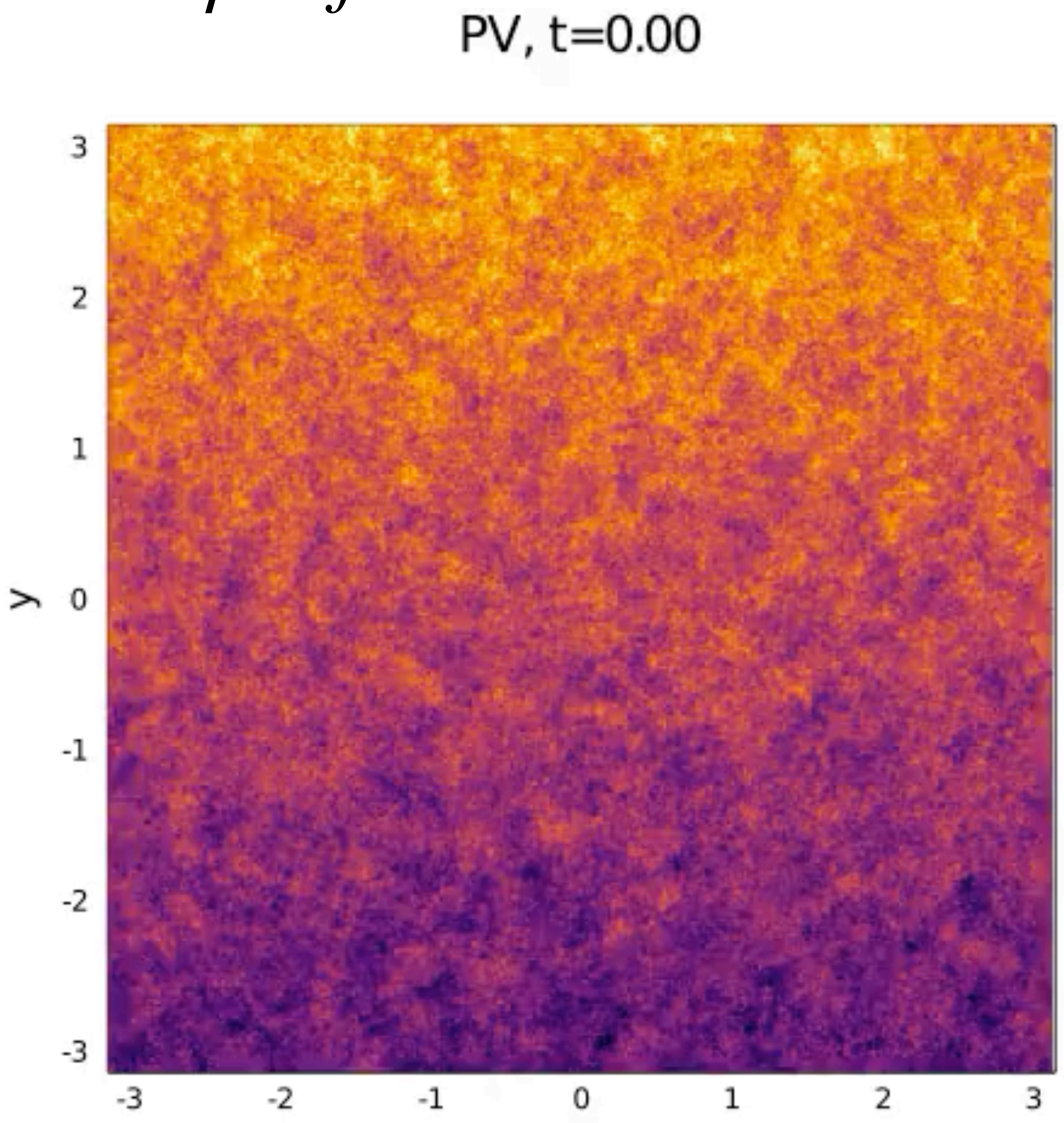
$f = 0$

rotating

$\nabla^2\psi$



$\nabla^2\psi + f$



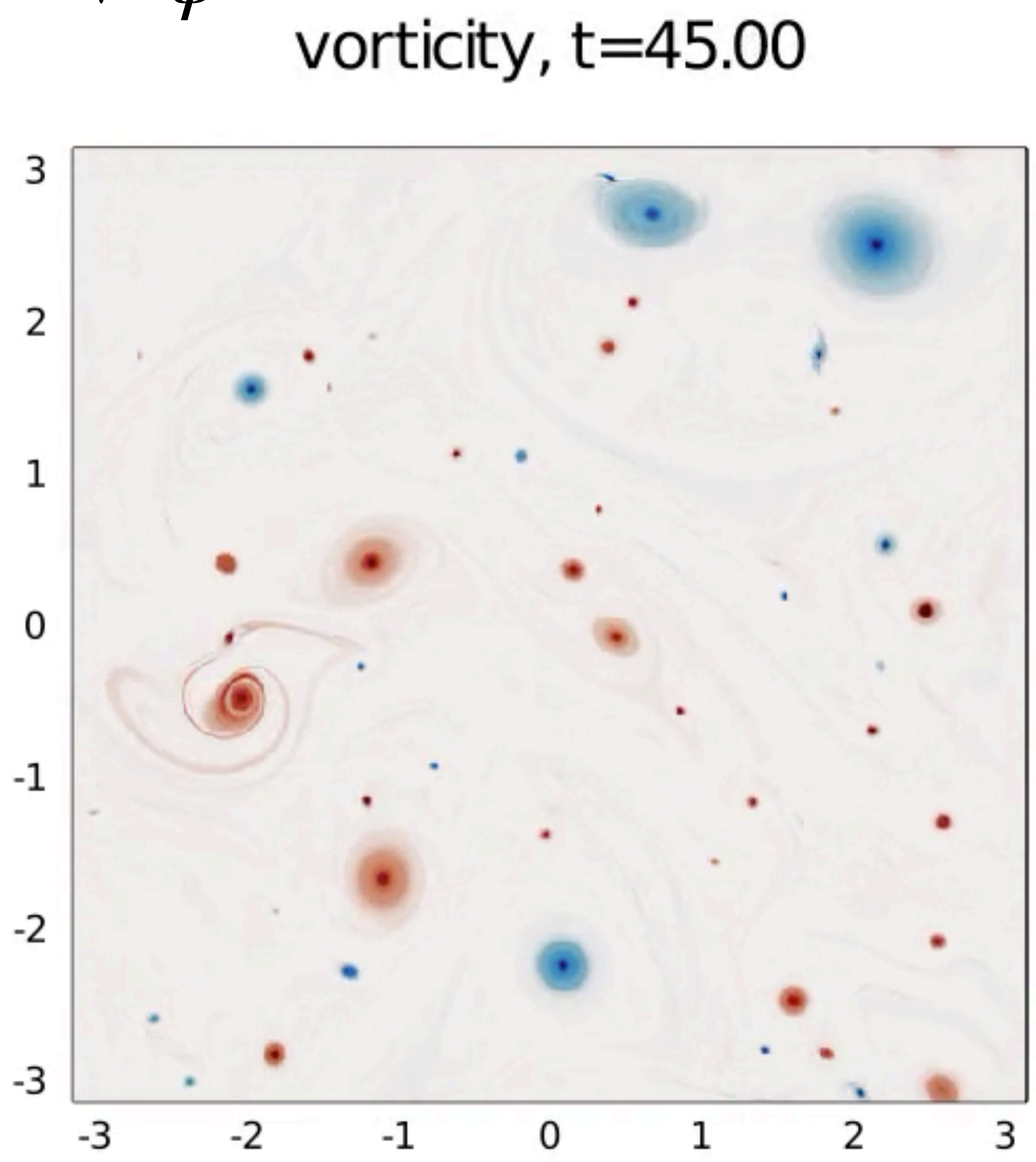
$f = f_0 + \beta y$

[simulations using GeophysicFlows.jl]

Quasi-Geostrophy **with Earth's curvature (β-plane)**

non-rotating

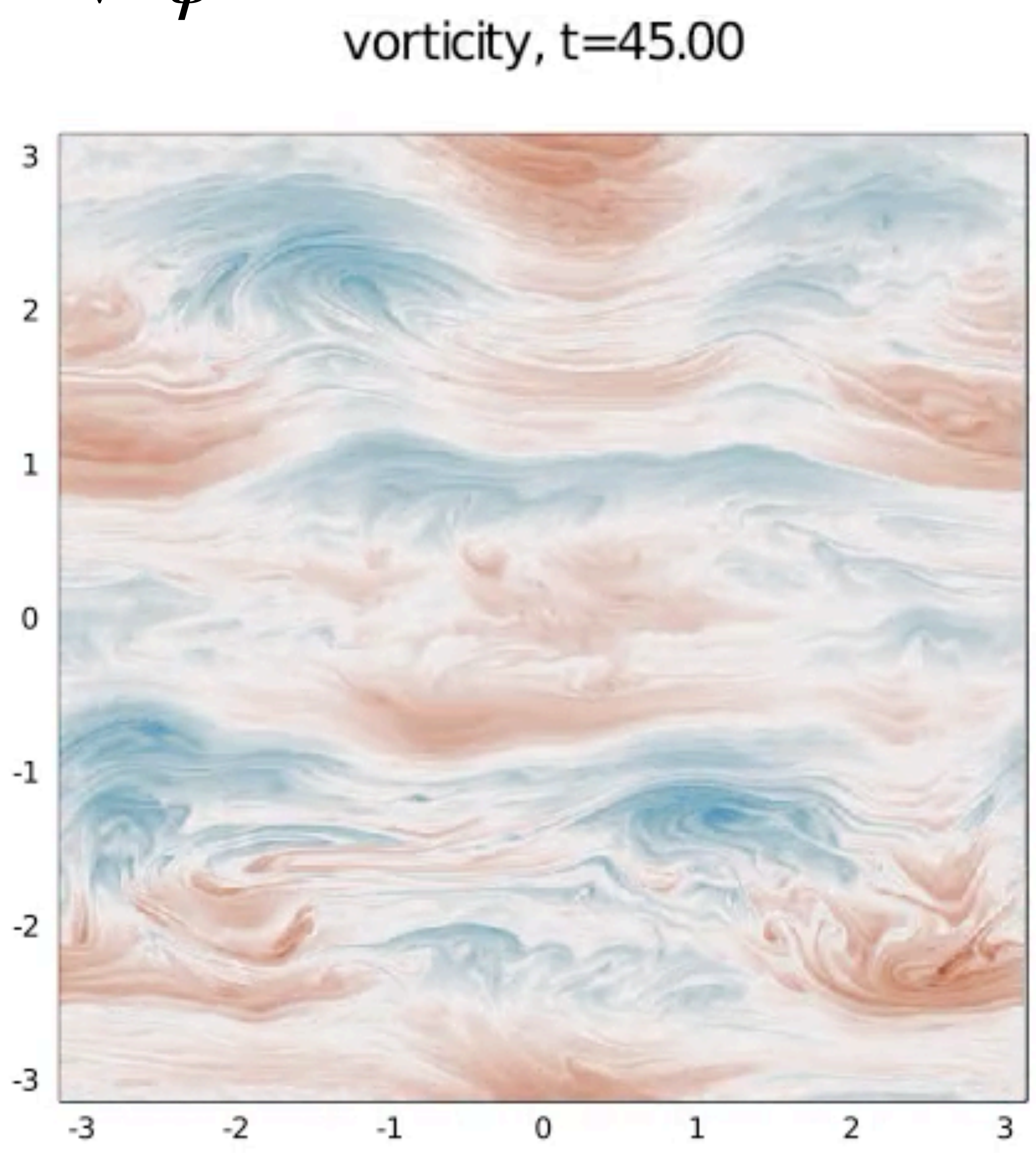
$\nabla^2\psi$



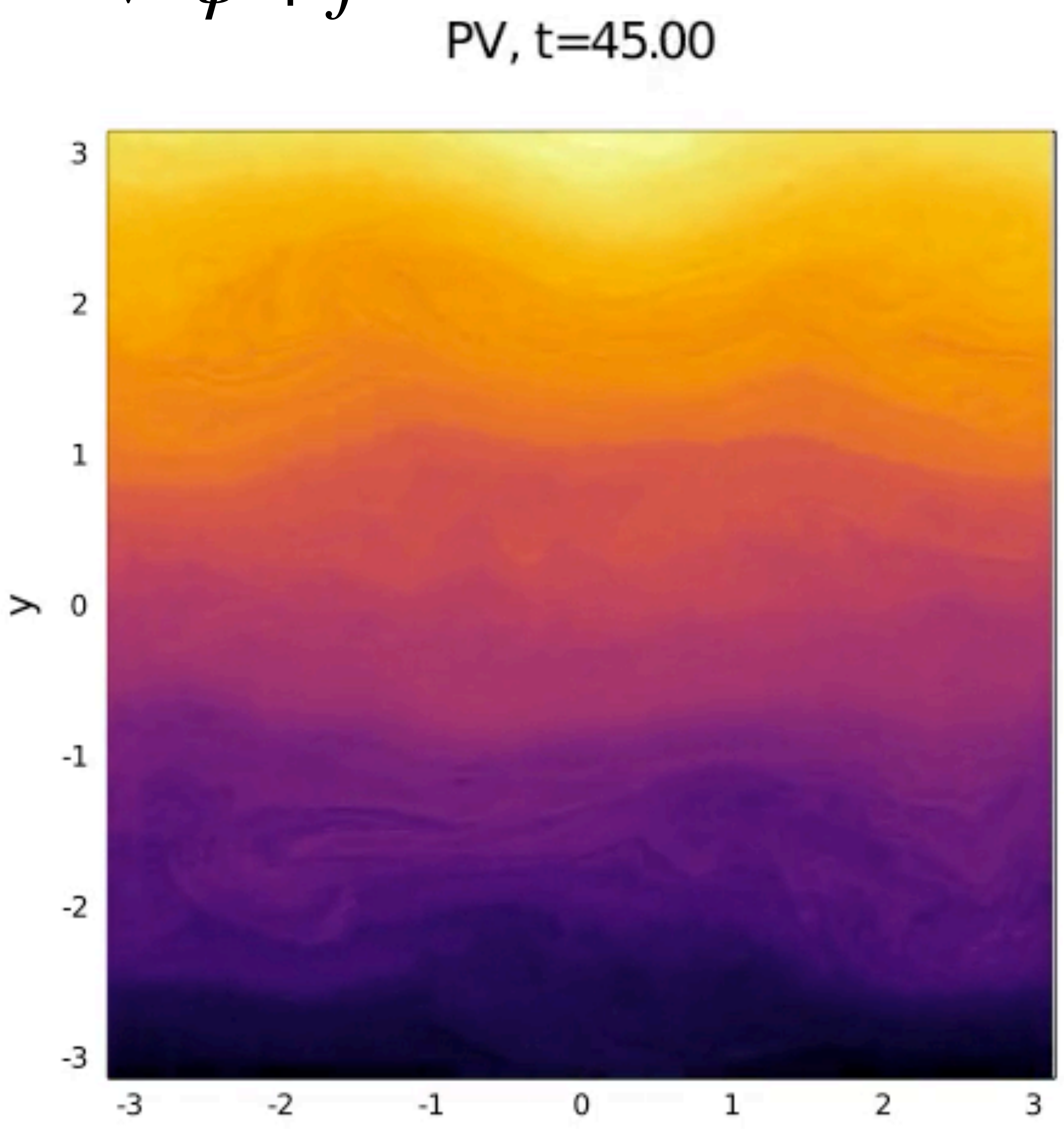
$f = 0$

rotating

$\nabla^2\psi$



$\nabla^2\psi + f$



$f = f_0 + \beta y$

[simulations using GeophysicFlows.jl]

Atmosphere & Ocean Dynamics
CLEx X School 2021 (?)

X = {Winter, Summer, Autum, Spring, Xmas, Easter, …}

THE FOLLOWING **PREVIEW** HAS BEEN APPROVED FOR

ALL AUDIENCES

BY THE MOTION PICTURE ASSOCIATION OF ~~AMERICA~~ ARC, INC.

Jupyter notebooks for reproducing animations can be found at:

github.com/navidcy/CLExWinterSchool2020